Construction techniques for cubical complexes, odd cubical 4-polytopes, and prescribed dual manifolds

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Abstract

We provide a number of new construction techniques for cubical complexes and cubical polytopes, and thus for cubifications (hexahedral mesh generation). As an application we obtain an instance of a cubical 4-polytope that has a non-orientable dual manifold (a Klein bottle). This confirms an existence conjecture of Hetyei (1995).

More systematically, we prove that every normal crossing codimension one immersion of a compact 2-manifold into $\mathbb{R}^3$ is PL-equivalent to a dual manifold immersion of a cubical 4-polytope. As an instance we obtain a cubical 4-polytope with a cubation of Boy's surface as a dual manifold immersion, and with an odd number of facets. Our explicit example has 17 718 vertices and 16 533 facets. Thus we get a parity changing operation for 3-dimensional cubical complexes (hexa meshes); this solves problems of Eppstein, Thurston, and others.

Keywords: Cubical complexes, cubical polytopes, regular subdivisions, normal crossing codimension one PL immersions, construction techniques, cubical meshes, Boy's surface

MSC 2000 Subject Classification: 52B12, 52B11, 52B05, 57Q05

1 Introduction

A d-polytope is *cubical* if all its proper faces are combinatorial cubes, that is, if each k-face of the polytope, $k \in \{0, \dots, d-1\}$ is combinatorially equivalent to the k-dimensional standard cube.

It has been observed by Stanley, MacPherson, and others (cf. [3] [20]) that every cubical d-polytope P determines a PL immersion of an abstract cubical $(d-2)$-manifold into the polytope

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boundary $\partial P \cong S^{d-1}$. The immersed manifold is orientable if and only if the 2-skeleton of the cubical d-polytope ($d \geq 3$) is "edge orientable" in the sense of Hetyei, who conjectured that there are cubical 4-polytopes that are not edge-orientable [17, Conj. 2].

In the more general setting of cubical PL $(d-1)$-spheres, Babson and Chan [3] have observed that *every* type of normal crossing PL immersion of a $(d-2)$-manifold into a $(d-1)$-sphere appears among the dual manifolds of some cubical PL $(d-1)$-sphere.

No similarly general result is available for cubical polytopes. The reason for this may be traced/blamed to a lack of flexible construction techniques for cubical polytopes, and more generally, for cubical complexes (such as the "hexahedral meshes" that are of great interest in CAD and in Numerical Analysis).

In this paper, we develop a number of new and improved construction techniques for cubical polytopes. We try to demonstrate that it always pays off to carry along convex lifting functions of high symmetry. The most complicated and subtle element of our constructions is the "generalized regular Hexhoop" of Section 6.4, which yields a cubification of a d-polytope with a hyperplane of symmetry, where a (suitable) lifting function may be specified on the boundary.

Our work is extended by the first author in [28], where additional construction techniques for *cubifications* (i. e. cubical subdivisions of d-polytopes with prescribed boundary subdivisions) are discussed.

Using the constructions developed here, we achieve the following constructions and results:

- A rather simple construction yields a cubical 4-polytope (with 72 vertices and 62 facets) for which the immersed dual 2-manifold is not orientable: One of its components is a Klein bottle. Apparently this is the first example of a cubical polytope with a non-orientable dual manifold. Its existence confirms a conjecture of Hetyei (Section 5).
- More generally, all PL-types of normal crossing immersions of 2-manifolds appear as dual manifolds in the boundary complexes of cubical 4-polytopes (Section 7). In the case of non-orientable 2-manifolds of odd genus, this yields cubical 4-polytopes with an odd number of facets. From this, we also obtain a complete characterization of the lattice of f-vectors of cubical 4-polytopes (Section 9).
- In particular, we construct an explicit example with 17 718 vertices and 16 533 facets of a cubical 4-polytope which has a cubation of Boy's surface (projective plane with exactly one triple point) as a dual manifold immersion (Section 8).
- Via Schlegel diagrams, this implies that every 3-cube has a cubical subdivision into an even number of cubes that does not subdivide the boundary complex. Thus for every cubification of a 3-dimensional domain there is also a cubification of the opposite parity (Section 10). This answers questions by Bern, Eppstein, Erickson, and Thurston [5] [10] [31].

2 Basics

For the following we assume that the readers are familiar with the basic combinatorics and geometry of convex polytopes. In particular, we will be dealing with cubical polytopes (see Grünbaum [15, Sect. 4.6]), polytopal (e. g. cubical) complexes, regular subdivisions (see Ziegler [33, Sect. 5.1]), and Schlegel diagrams [15, Sect. 3.3] [33, Sect. 5.2]. For cell complexes, barycentric subdivision and related notions we refer to Munkres [24]. Suitable references for the basic concepts about PL manifolds, embeddings and (normal crossing) immersions include Hudson [19] and Rourke & Sanderson [27].

2.1 Almost cubical polytopes

All proper faces of a cubical d-polytope have to be combinatorial cubes. We define an *almost cubical* d-polytope as a pair (P, F), where F is a specified facet of P such that all facets of P other than F are required to be combinatorial cubes. Thus, F need not be a cube, but it will be cubical.

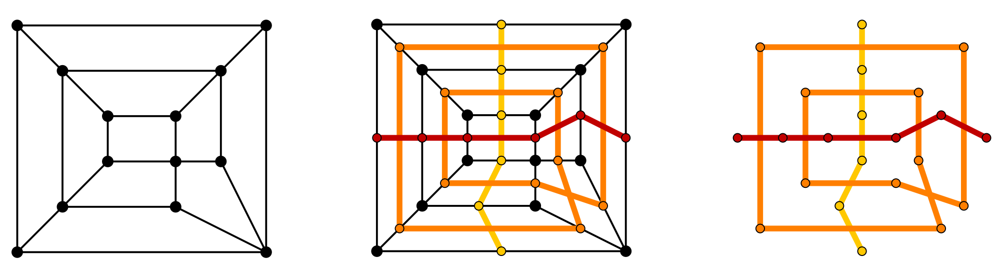
By $\mathcal{C}(P)$ we denote the polytopal complex given by a polytope P and all its faces. By $\mathcal{C}(\partial P)$ we denote the *boundary complex* of P, consisting of all proper faces of P. If P is a cubical polytope, then $\mathcal{C}(\partial P)$ is a cubical complex. If (P, F) is almost cubical, then the *Schlegel complex* $\mathcal{C}(\partial P)\backslash\{F\}$ is a cubical complex that is combinatorially isomorphic to the Schlegel diagram $\textsc{Schlegel}(P, F)$ of P based on F.

2.2 Cubifications

A *cubification* of a cubical PL $(d-1)$-sphere $\mathcal{S}^{d-1}$ is a cubical d-ball $\mathcal{B}^d$ with boundary $\mathcal{S}^{d-1}$. A double counting argument shows that every cubical $(d-1)$-sphere that admits a cubification has an even number of facets. Whether this condition is sufficient is a challenging open problem, even for $d = 3$ (compare [5], [10]).

2.3 Dual manifolds

For every (pure) cubical d-dimensional complex $\mathcal{C}$, $d > 1$, the *derivative complex* is an abstract cubical cell $(d-1)$-dimensional complex $\mathcal{D}(\mathcal{C})$ whose vertices may be identified with the edge midpoints of the complex, while the facets "separate the opposite facets of a facet of $\mathcal{C}$," that is, they correspond to pairs $(F, [e])$, where F is a facet of $\mathcal{C}$ and $[e]$ denotes a "parallel class" of edges of F. This is a cell complex with $f_1(\mathcal{C})$ vertices and $(d-1)f_{d-1}(\mathcal{C})$ cubical facets of dimension $d-1$, $d-1$ of them for each facet of $\mathcal{C}$. Hence the derivative complex $\mathcal{D}(\mathcal{C})$ is pure $(d-1)$-dimensional. See Babson & Chan [3, Sect. 4].



(a) $\mathcal{C}$ (b) $\mathcal{C}$ and $\mathcal{D}(\mathcal{C})$ (c) $\mathcal{D}(\mathcal{C})$

Figure 1: The derivative complex of a cubical 2-complex $\mathcal{C}$.

In the case of cubical PL spheres (for instance boundary complexes of cubical polytopes), or cubical PL balls, the derivative complex is a (not necessarily connected) manifold, and we call each connected component of the derivative complex $\mathcal{D}(P)$ of a cubical complex $\mathcal{C}$ a *dual manifold* of $\mathcal{C}$. If the cubical complex $\mathcal{C}$ is a sphere then the dual manifolds of $\mathcal{C}$ are manifolds without boundary. If $\mathcal{C}$ is a ball, then some (possibly all) dual manifolds have non-empty boundary components, namely the dual manifolds of $\partial\mathcal{C}$.

The derivative complex, and thus each dual manifold, comes with a canonical immersion into the boundary of P. More precisely, the barycentric subdivision of $\mathcal{D}(P)$ has a simplicial map to the barycentric subdivision of the boundary complex ∂P, which is a codimension one normal crossing immersion into the simplicial sphere $\mathrm{sd}\,(\mathcal{C}(\partial P))$. (*Normal crossing* means that each multiple-intersection point is of degree $k \leq d$ and there is a neighborhood of each multiple intersection point that is PL isomorphic to (a neighborhood of) a point which is contained in k pairwise perpendicular hyperplanes.)

Restricted to a dual manifold, this immersion may be an embedding or not.

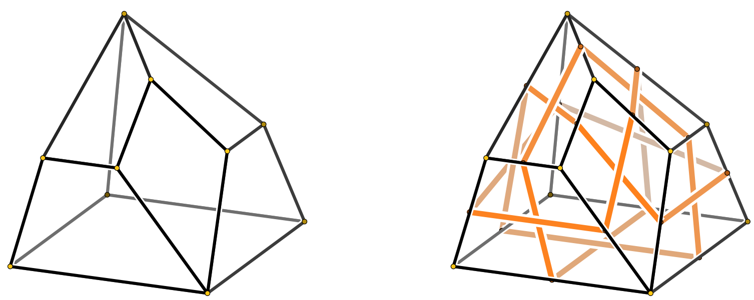


FIGURE 2: The cubical octahedron O_8 (the only combinatorial type of a cubical 3-polytope with 8 facets), and its single immersed dual manifold.

In the case of cubical 3-polytopes, the derivative complex may consist of one or many 1-spheres. For example, for the 3-cube it consists of three 1-spheres, while for the "cubical octahedron" O_8 displayed in Figure 2 the dual manifold is a single immersed S^1 (with 8 double points).

In the case of 4-polytopes, the dual manifolds are surfaces (compact 2-manifolds without boundary). As an example, we here display a Schlegel diagram of a "neighborly cubical" 4-polytope (with the graph of the 5-cube), with f-vector $(32, 80, 96, 48)$.

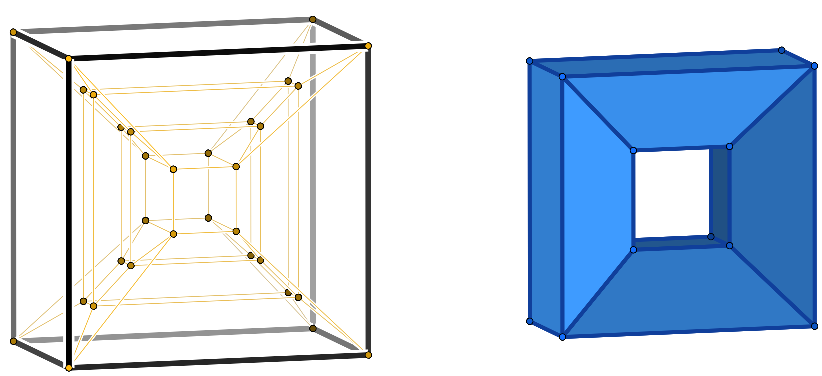


FIGURE 3: A Schlegel diagram of the "neighborly cubical" 4-polytope C_4^5 with the graph of the 5-cube, and its dual torus. All other dual manifolds are embedded 2-spheres.

According to Joswig & Ziegler [21] this may be constructed as

$$C_4^5 \quad := \quad \mathrm{conv}((Q \times 2Q) \; \cup \; (2Q \times Q)), \qquad \text{where } Q = [-1, +1]^2.$$

Here the dual manifolds are four embedded cubical 2-spheres S^2 with f-vector $(16, 28, 14)$ — of two different combinatorial types — and one embedded torus T with f-vector $(16, 32, 16)$.

2.4 Orientability

Let P be a cubical d-polytope ($d \geq 3$). The immersed dual manifolds in its boundary cross the edges of the polytope transversally.

Thus we find that orientability of the dual manifolds is equivalent to the possibility to give *consistent* edge orientations to the edges of the P, that is, in each 2-face of P opposite edges should get parallel (rather than antiparallel) orientations; compare Hetyei [17]. Figure 4 shows such an edge orientation for a cubical 3-polytope (whose derivative complex consists of three circles, so it has 8 consistent edge orientations in total).

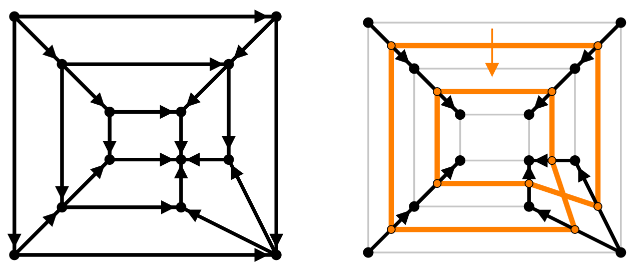


FIGURE 4: Edge-orientation of (the Schlegel diagram of) a cubical 3-polytope. The edges marked on the right must be oriented consistently.

One can attempt to obtain such edge orientations by moving to from edge to edge across 2-faces. The obstruction to this arises if on a path moving from edge to edge across quadrilateral 2-faces we return to an already visited edge, with reversed orientation, that is, if we close a *cubical Möbius strip* with parallel inner edges, as displayed in the figure. (Such an immersion is not necessarily embedded, that is, some 2-face may be used twice for the Möbius strip.)

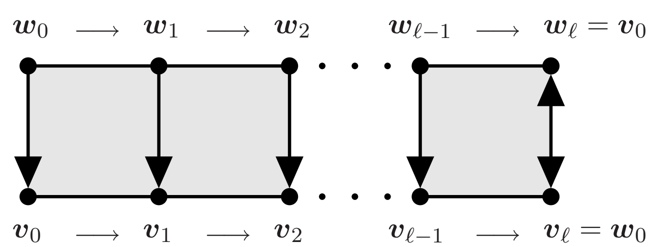


FIGURE 5: A cubical Möbius strip with parallel inner edges.

Proposition 2.1. *For every cubical d-polytope ($d \geq 3$), the following are equivalent:*

- *All dual manifolds of P are orientable.*
- *The 2-skeleton of P has a consistent edge orientation.*
- *The 2-skeleton of P contains no immersion of a cubical Möbius strip with parallel inner edges.*

2.5 From PL immersions to cubical PL spheres

The emphasis in this paper is on cubical convex d-polytopes. In the more general setting of cubical PL $(d-1)$-spheres, one has more flexible tools available. In this setting, Babson & Chan [3] proved that "all PL codimension 1 normal crossing immersions appear." The following sketch is meant to explain the Babson-Chan theorem geometrically (it is presented in a combinatorial framework and terminology in [3]), and to briefly indicate which parts of their construction are available in the polytope world.

Construction 1: BABSON-CHAN [3]

Input: A normal crossing immersion $j : \mathcal{M}^{d-2} \to \mathcal{S}^{d-1}$ of a triangulated PL manifold $\mathcal{M}^{d-2}$ of dimension $d-2$ into a PL simplicial $(d-1)$-sphere.

Output: A cubical PL $(d-1)$-sphere with a dual manifold immersion PL-equivalent to j.

(1) Perform a barycentric subdivision on $\mathcal{M}^{d-2}$ and $\mathcal{S}^{d-1}$.

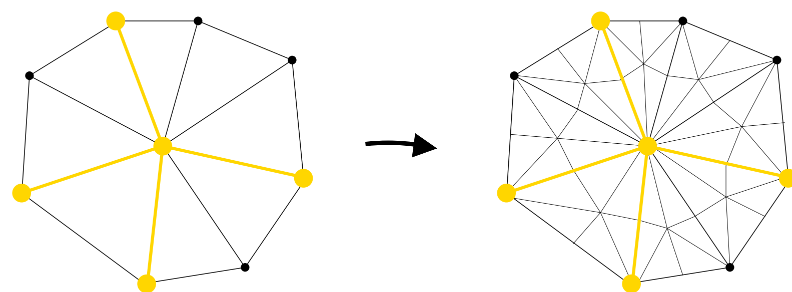


FIGURE 6: Step 1. Performing a barycentric subdivision. (We illustrate the impact of the construction on 2-ball, which might be part of the boundary of a 2-sphere. The immersion which is shown in bold has a single double-intersection point.)

(Here each i-simplex is replaced by $(i+1)!$ new i-simplices, which is an even number for $i > 0$. This step is done only to ensure parity conditions on the f-vector, especially that the number of facets of the final cubical sphere is congruent to the Euler characteristic of $\mathcal{M}^{d-2}$. Barycentric subdivisions are easily performed in the polytopal category as well, see Ewald & Shephard [11].)

(2) Perform a "cubical barycentric subdivision" on $\mathcal{M}^{d-2}$ and $\mathcal{S}^{d-1}$.

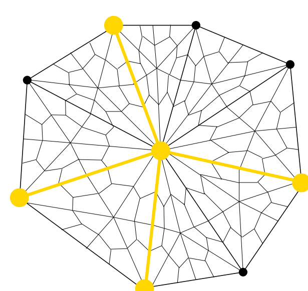


FIGURE 7: Step 2. Performing a cubical barycentric subdivision.

(This is the standard tool for passage from a simplicial complex to a PL-homeomorphic cubical complex; here every i-simplex is subdivided into $i+1$ different i-cubes. Such cubations can be performed in the polytopal category according to Shephard [29]: If the starting triangulation of $\mathcal{S}^{d-1}$ was polytopal, the resulting cubation will be polytopal as well.)

(3) "Thicken" the cubical $(d-1)$-sphere along the immersed $(d-2)$-manifold, to obtain the cubical $(d-1)$-sphere $BC(\mathcal{S}^{d-1}, j(\mathcal{M}^{d-2}))$.
(In this step, every $(d-1-i)$-cube in the i-fold multiple point locus results in a new $(d-1)$-cube. The original immersed manifold, in its cubified subdivided version, now

appears as a dual manifold in the newly resulting $(d-1)$-cubes. This last step is the one that seems hard to perform for polytopes in any non-trivial instance.)

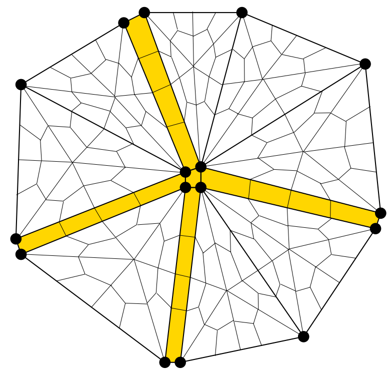


FIGURE 8: The outcome of the Babson-Chan construction: A cubical sphere with a dual manifold immersion that is PL-equivalent to the input immersion j.

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3 Lifting polytopal subdivisions

3.1 Regular balls

In the following, the primary object we deal with is a *regular ball*: a regular polytopal subdivision $\mathcal{B}$ of a convex polytope $P = |\mathcal{B}|$.

Definition 3.1 (regular subdivision, lifting function). A polytopal subdivision $\mathcal{B}$ is *regular* (also known as *coherent* or *projective*) if it admits a *lifting function*, that is, a concave function $f : |P| \to \mathbb{R}$ whose domains of linearity are the facets of the subdivision. (A function $g : D \to \mathbb{R}$ is *concave* if for all $\boldsymbol{x}, \boldsymbol{y} \in D$ and $0 < \lambda < 1$ we have $g(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \lambda g(\boldsymbol{x}) + (1-\lambda)g(\boldsymbol{y})$.)

In this definition, subdivisions of the boundary are allowed, that is, we do not necessarily require that the faces of $P = |\mathcal{B}|$ are themselves faces in $\mathcal{B}$.

In the sequel we focus on regular *cubical* balls. Only in some cases we consider regular non-cubical balls.

Example. If (P, F) is an almost cubical polytope, then the Schlegel diagram based on F, which we denote by SCHLEGEL(P, F), is a regular cubical ball (without subdivision of the boundary).

Lemma 3.2. *If $\mathcal{B}$ is a regular cubical d-ball, then there is a regular cubical ball $\mathcal{B}'$ without subdivision of the boundary, combinatorially isomorphic to $\mathcal{B}$.*

Proof. Using a positive lifting function $f : |\mathcal{B}| \to \mathbb{R}$, the d-ball $\mathcal{B}$ may be lifted to $\widetilde{\mathcal{B}}$ in $\mathbb{R}^{d+1}$, by mapping each $\boldsymbol{x} \in |\mathcal{B}|$ to $(\boldsymbol{x}, f(\boldsymbol{x})) \in \mathbb{R}^{d+1}$.

Viewed from $\boldsymbol{p} := \lambda\boldsymbol{e}_{d+1}$ for sufficiently large λ, this lifted ball will appear to be *strictly convex*, that is, its boundary is a convex polytope (rather that a boundary subdivision of a convex polytope). Thus one may look at the polytopal complex that consists of the cones spanned by faces of $\widetilde{\mathcal{B}}$ with apex $\boldsymbol{p}$. This polytopal complex is regular, since it appears convex when viewed from $\boldsymbol{p}$, which yields a lifting function for the restriction of $\widetilde{\mathcal{B}}$ to the hyperplane given by $x_{d+1} = 0$, which may be taken to be $\mathcal{B}'$. ☐

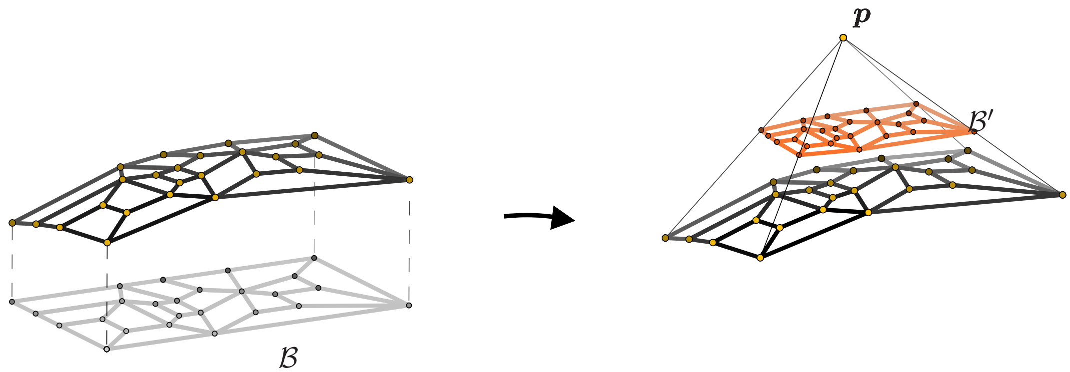


FIGURE 9: Illustration of the 'convexification' of a regular ball (Lemma 3.2).

3.2 Lifted balls

When constructing cubical complexes we often deal with regular cubical balls which are equipped with a lifting function. A *lifted d-ball* is a pair $(\mathcal{B}, h)$ consisting of a regular d-ball $\mathcal{B}$ and a lifting function h of $\mathcal{B}$. The *lifted boundary* of a lifted ball $(\mathcal{B}, h)$ is the pair $(\partial\mathcal{B}, h|_{\partial\mathcal{B}})$.

If $(\mathcal{B}, h)$ is a lifted d-ball in $\mathbb{R}^{d'}$ then $\mathrm{lift}(\mathcal{B}, h)$ denotes the copy of $\mathcal{B}$ in $\mathbb{R}^{d'+1}$ with vertices $(\boldsymbol{v}, h(\boldsymbol{v})) \in \mathbb{R}^{d'+1}$, $\boldsymbol{v} \in \mathrm{vert}(\mathcal{B})$. (In the sequel we sometimes do not distinguish between these two interpretations of a lifted ball.) We rely on Figure 10 for the illustration of this correspondence.

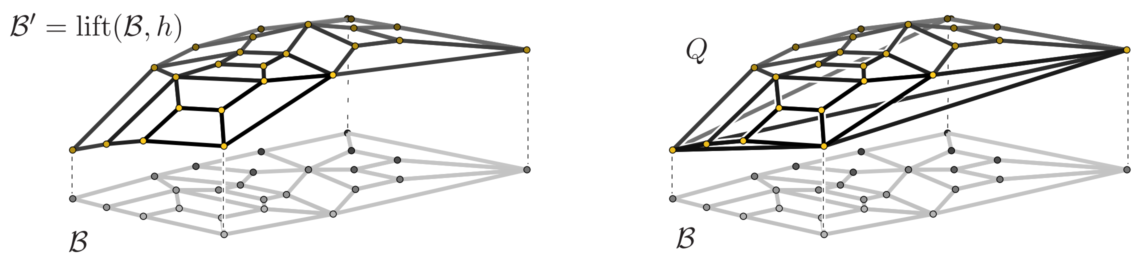


FIGURE 10: A lifted cubical ball $(\mathcal{B}, h)$ and its lifted copy $\mathrm{lift}(\mathcal{B}, h)$. The figure on the right shows the convex hull $Q = \mathrm{conv}(\mathrm{lift}(\mathcal{B}, h))$.

Notation. We identify $\mathbb{R}^d$ with $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$, and decompose a point $\boldsymbol{x} \in \mathbb{R}^{d+1}$ as $\boldsymbol{x} = (\pi(\boldsymbol{x}), \gamma(\boldsymbol{x}))$, where $\gamma(\boldsymbol{x})$ is the last coordinate of $\boldsymbol{x}$ and $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is the projection that eliminates the last coordinate.

Often a lifted ball $(\mathcal{B}, \psi)$ is constructed as follows: Let P be a d-polytope (in $\mathbb{R}^d$) and $Q \subset \mathbb{R}^{d+1}$ a $(d+1)$-polytope such that $\pi(Q) = P$. Then the complex $\mathcal{B}'$ given as the set of upper faces of Q determines a lifted polytopal subdivision $(\mathcal{B}, \psi)$ of P (where $\mathcal{B} := \pi(\mathcal{B}')$ and ψ is determined the vertex heights $\gamma(\boldsymbol{v})$, $\boldsymbol{v} \in \mathrm{vert}(\mathcal{B}')$). Hence $\mathrm{lift}(\mathcal{B}, \psi)$ equals $\mathcal{B}'$. Compare again Figure 10.

The *lifted boundary subdivision* of a d-polytope P is a pair $(\mathcal{S}^{d-1}, \psi)$ consisting of a polytopal subdivision $\mathcal{S}^{d-1}$ of the boundary of P and a piece-wise linear function $\psi : |\partial P| \to \mathbb{R}$ such that for each facet F of P the restriction of ψ to F is a lifting function of the induced subdivision $\mathcal{S}^{d-1} \cap F$ of F.

3.3 The patching lemma

Often regular cubical balls are constructed from other regular balls. The following "patching lemma", which appears frequently in the construction of regular subdivisions (see [22, Cor. 1.12] or [8, Lemma 3.2.2]) is a basic tool for this.

Notation. For a d-polytope $P \subset \mathbb{R}^{d'}$, a polytopal subdivision $\mathcal{T}$ of P and a hyperplane H in $\mathbb{R}^{d'}$, we denote by $\mathcal{T} \cap H$ the *restriction* of $\mathcal{T}$ to H, which is given by

$$\mathcal{T} \cap H := \{F \cap H \,:\, F \in \mathcal{T}\}\,.$$

For two d-polytopes P, Q with $Q \subset P$ and a polytopal subdivision $\mathcal{T}$ of P we denote by $\mathcal{T} \cap Q$ the *restriction* of $\mathcal{T}$ to Q, which is given by

$$\mathcal{T} \cap Q := \{F \cap Q \,:\, F \in \mathcal{T}\}\,.$$

By $\mathrm{fac}(\mathcal{S})$ we denote the set of facets of a complex $\mathcal{S}$.

Lemma 3.3 ("Patching lemma"). *Let Q be a d-polytope. Assume we are given the following data:*

- ▷ *A regular polytopal subdivision $\mathcal{S}$ of Q (the "raw subdivision").*
- ▷ *For each facet F of $\mathcal{S}$, a regular polytopal subdivision $\mathcal{T}_F$ of F, such that $\mathcal{T}_F \cap F' = \mathcal{T}_{F'} \cap F$ for all facets F, F' of $\mathcal{S}$.*
- ▷ *For each facet F of $\mathcal{S}$, a concave lifting function h_F of $\mathcal{T}_F$, such that $h_F(\boldsymbol{x}) = h_{F'}(\boldsymbol{x})$ for all $\boldsymbol{x} \in F \cap F'$, where F, F' are facets of $\mathcal{S}$.*

Then this uniquely determines a regular polytopal subdivision $\mathcal{U} = \bigcup_F \mathcal{T}_F$ of Q (the "fine subdivision"). Furthermore, for every lifting function g of $\mathcal{S}$ there exists a small $\varepsilon_0 > 0$ such that for all ε in the range $\varepsilon_0 > \varepsilon > 0$ the function $g + \varepsilon h$ is a lifting function of $\mathcal{U}$, where h is the piecewise linear function $h : |Q| \to \mathbb{R}$ which on each $F \in \mathcal{S}$ is given by h_F.

Proof. Let g be a lifting function of $\mathcal{S}$. For a parameter $\varepsilon > 0$ we define a piece-wise linear function $\phi_\varepsilon : |P| \to \mathbb{R}$ that on $\boldsymbol{x} \in F \in \mathrm{fac}(\mathcal{S})$ takes the value $\phi_\varepsilon(\boldsymbol{x}) = g(\boldsymbol{x}) + \varepsilon h_F(\boldsymbol{x})$. (It is well-defined since the h_F coincide on the ridges of $\mathcal{S}$.) The domains of linearity of ϕ_ε are given by the facets of the "fine" subdivision $\mathcal{U}$. If ε tends to zero then ϕ_ε tends to the concave function g. This implies that there exists a small $\varepsilon_0 > 0$ such that ϕ_ε is concave and thus a lifting function of $\mathcal{U}$, for $\varepsilon_0 > \varepsilon > 0$. □

3.4 Products and prisms

Lemma 3.4 ("Product lemma"). *Let $(\mathcal{B}_1, h_1)$ be a lifted cubical d_1-ball in $\mathbb{R}^{d'_1}$ and $(\mathcal{B}_2, h_2)$ be a lifted cubical d_2-ball in $\mathbb{R}^{d'_2}$.*
Then the product $\mathcal{B}_1 \times \mathcal{B}_2$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ is a regular cubical $(d_1 + d_2)$-ball in $\mathbb{R}^{d'_1 + d'_2}$.

Proof. Each cell of $\mathcal{B}_1 \times \mathcal{B}_2$ is a product of two cubes. Hence $\mathcal{B}_1 \times \mathcal{B}_2$ is a cubical complex. A lifting function h of $\mathcal{B}_1 \times \mathcal{B}_2$ is given by the sum of h_1 and h_2, that is, by $h((\boldsymbol{x}, \boldsymbol{y})) := h_1(\boldsymbol{x}) + h_2(\boldsymbol{y})$, for $\boldsymbol{x} \in |\mathcal{B}_1|, \boldsymbol{y} \in |\mathcal{B}_2|$. □

As a consequence, the *prism* $\mathrm{prism}(\mathcal{C})$ over a cubical d-complex $\mathcal{C}$ yields a cubical $(d + 1)$-dimensional complex. Furthermore, the prism over a regular cubical ball $\mathcal{B}$ yields a regular cubical $(d + 1)$-ball.

3.5 Piles of cubes

For integers $\ell_1, \dots, \ell_d \geq 1$, the *pile of cubes* $P_d(\ell_1, \dots, \ell_d)$ is the cubical d-ball formed by all unit cubes with integer vertices in the d-polytope $P := [0, \ell_1] \times \ldots \times [0, \ell_d]$, that is, the cubical d-ball formed by the set of all d-cubes

$$C(k_1, \dots, k_d) := [k_1, k_1 + 1] \times \cdots \times [k_d, k_d + 1]$$

for integers $0 \leq k_i < \ell_i$ together with their faces [33, Sect. 5.1].

The pile of cubes $P_d(\ell_1, \dots, \ell_d)$ is a product of 1-dimensional subdivisions, which are regular. Hence the product lemma implies that $P_d(\ell_1, \dots, \ell_d)$ is a regular cubical subdivision of the d-polytope P.

3.6 Connector polytope

The following construction yields a “connector” polytope that may be used to attach cubical 4-polytopes resp. regular cubical 4-balls without the requirement that the attaching facets are projectively equivalent.

Lemma 3.5. *For any combinatorial* 3*-cube* F *there is a combinatorial* 4*-cube* C *that has both (a projective copy of)* F *and a regular* 3*-cube* F' *as (adjacent) facets.*

Proof. After a suitable projective transformation we may assume that $F \subset \mathbb{R}^3$ has a unit square Q as a face. Now the prism $F \times I$ over F has F and $Q \times I$ as adjacent facets, where the latter is a unit cube. □

4 Basic construction techniques

4.1 Lifted prisms

While there appears to be no simple construction that would produce a cubical $(d+1)$-polytope from a given cubical d-polytope, we do have a simple prism construction that produces regular cubical $(d+1)$-balls from regular cubical d-balls.

Construction 2: Lifted prism

Input: A lifted cubical d-ball $(\mathcal{B}, h)$.
Output: A lifted cubical $(d+1)$-ball $\text{LiftedPrism}(\mathcal{B}, h)$
which is combinatorially isomorphic to the prism over $\mathcal{B}$.

We may assume that the convex lifting function h defined on $P := |\mathcal{B}|$ is strictly positive. Then the lifted facets of $\text{LiftedPrism}(\mathcal{B}, h)$ may be taken to be the sets

$$\widetilde{F} \quad := \quad \{(\boldsymbol{x}, t, h(\boldsymbol{x})) : \boldsymbol{x} \in F,\ -h(\boldsymbol{x}) \leq t \leq +h(\boldsymbol{x})\}, \qquad F \in \mathrm{fac}(\mathcal{B}).$$

If $\mathcal{B}$ does not subdivide the boundary of P, then $\text{LiftedPrism}(\mathcal{B}, h)$ does not subdivide the boundary of $|\,\text{LiftedPrism}(\mathcal{B}, h)|$. In this case $\widehat{P} := |\,\text{LiftedPrism}(\mathcal{B}, h)|$ is a cubical $(d+1)$-polytope whose boundary complex is combinatorially isomorphic to the boundary of the prism

over $\mathcal{B}$. The f-vector of $\widehat{P}$ is then given by

$$f_k(\widehat{P}) \quad = \quad \begin{cases} 2f_0(\mathcal{B}) & \text{for } k = 0, \\ 2f_k(\mathcal{B}) + f_{k-1}(\partial\mathcal{B}) & \text{for } 0 < k \le d. \end{cases}$$

Figure 11 shows the lifted prism over a lifted cubical 2-ball.

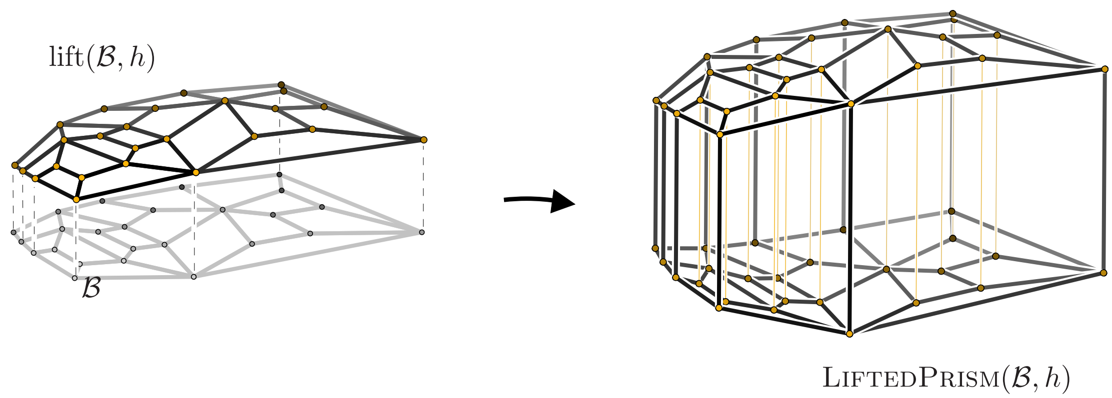


FIGURE 11: The lifted prism of a lifted cubical d-ball $(\mathcal{B}, h)$, displayed for $d = 2$. The result is a (regular) cubical $(d+1)$-ball that is combinatorially isomorphic to the prism over $\mathcal{B}$.

Proposition 4.1 (Dual manifolds). *Up to PL-homeomorphism, the cubical ball* LiftedPrism$(\mathcal{B}, h)$ *has the following dual manifolds:*

- $\mathcal{N} \times I$ *for each dual manifold* $\mathcal{N}$ *of* $\mathcal{B}$,
- *one* $(d-1)$*-sphere combinatorially isomorphic to* $\partial\mathcal{B}$. □

4.2 Lifted prisms over two balls

Another modification of this construction is to take two different lifted cubical balls $(\mathcal{B}_1, h_1)$ and $(\mathcal{B}_2, h_2)$ with the same lifted boundary complex (that is, $\partial\mathcal{B}_1 = \partial\mathcal{B}_2$ with $h_1(\boldsymbol{x}) = h_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in \partial\mathcal{B}_1 = \partial\mathcal{B}_2$) as input. In this case the outcome is a cubical $(d+1)$-polytope which may not even have a cubification.

Construction 3: Lifted prism over two balls

Input: Two lifted cubical d-balls $(\mathcal{B}_1, h_1)$ and $(\mathcal{B}_2, h_2)$ with the same lifted boundary.

Output: A cubical $(d+1)$-polytope LiftedPrism$((\mathcal{B}_1, h_1), (\mathcal{B}_2, h_2))$ with lifted copies of $\mathcal{B}_1$ and $\mathcal{B}_1$ in its boundary.

If both balls do not subdivide their boundaries, we set $\mathcal{B}'_k := \mathcal{B}_k$ and $h'_k := h_k$ for $k \in \{1, 2\}$. Otherwise we apply the construction of the proof of Lemma 3.2 simultaneously to both lifted cubical balls $(\mathcal{B}_1, h_1)$ and $(\mathcal{B}_2, h_2)$ to obtain two lifted cubical d-balls $(\mathcal{B}'_1, h'_1)$ and $(\mathcal{B}'_2, h'_2)$ with the same support $Q = |\mathcal{B}_1| = |\mathcal{B}_2|$ which do not subdivide the boundary of Q.

We can assume that h'_1, h'_2 are strictly positive. Then $\widehat{Q} :=$ LiftedPrism$((\mathcal{B}_1, h_1), (\mathcal{B}_2, h_2))$ is defined as the convex hull of the points in

$$\{(\boldsymbol{x}, +h'_1(\boldsymbol{x})) : \boldsymbol{x} \in |\mathcal{B}'_1|\} \;\cup\; \{(\boldsymbol{x}, -h'_2(\boldsymbol{x})) : \boldsymbol{x} \in |\mathcal{B}'_2|\},$$

Since $\mathcal{B}_1'$ and $\mathcal{B}_2'$ both do not subdivide their boundaries, each of their proper faces yields a face of $\widehat{Q}$. Furthermore, $\widehat{Q}$ is a cubical $(d+1)$-polytope whose f-vector is given by

$$f_k(\widehat{Q}) \quad = \quad \begin{cases} f_0(\mathcal{B}_1) + f_0(\mathcal{B}_2) & \text{for } k = 0, \\ f_k(\mathcal{B}_1) + f_k(\mathcal{B}_2) + f_{k-1}(\partial\mathcal{B}_1) & \text{for } 0 < k \leq d. \end{cases}$$

See Figure 12.

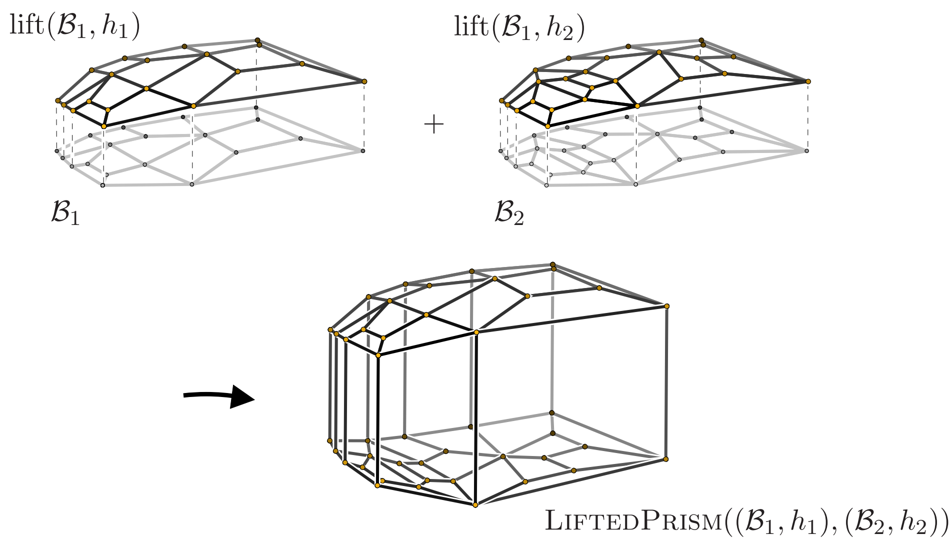


FIGURE 12: The lifted prism over two lifted cubical d-balls $(\mathcal{B}_1, h_1)$ and $(\mathcal{B}_2, h_2)$, displayed for $d = 2$. The outcome is a cubical $(d+1)$-polytope.

4.3 Schlegel caps

The following is a projective variant of the prism construction, applied to a d-polytope P.

Construction 4: SCHLEGEL CAP

Input: An almost cubical d-polytope (P, F_0)

Output: A regular cubical d-ball $\textsc{SchlegelCap}(P, F_0)$, with $P \subset |\textsc{SchlegelCap}(P, F_0)|$ which is combinatorially isomorphic to the prism over $\textsc{Schlegel}(P, F)$.

The construction of the Schlegel cap depends on two further pieces of input data, namely on a point $\boldsymbol{x}_0 \in \mathbb{R}^d$ beyond F_0 (and beneath all other facets of P; cf. [15, Sect. 5.2]) and on a hyperplane H that separates $\boldsymbol{x}_0$ from P. In terms of projective transformations it is obtained as follows:

(1) Apply a projective transformation that moves $\boldsymbol{x}_0$ to infinity while fixing H pointwise. This transformation moves the Schlegel complex $\mathcal{C}(\partial P)\backslash\{F_0\}$ to a new cubical complex $\mathcal{E}$.
(2) Reflect the image $\mathcal{E}$ of the Schlegel complex in H, and call its reflected copy $\mathcal{E}'$.
(3) Build the polytope bounded by $\mathcal{E}$ and $\mathcal{E}'$.
(4) Reverse the projective transformation of (1).

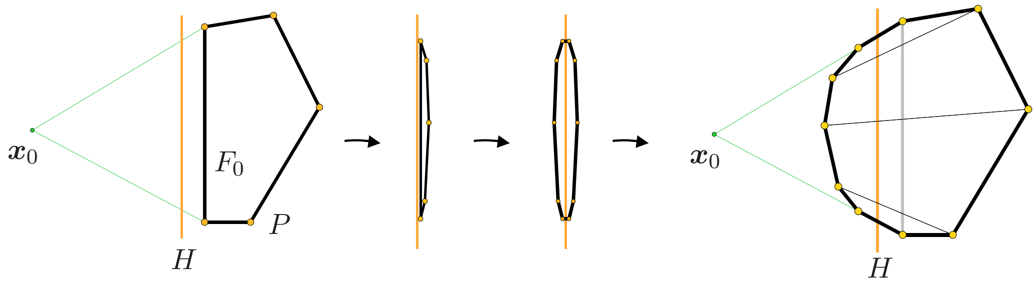


FIGURE 13: Construction steps of the Schlegel cap over an almost cubical polytope.

An alternative description, avoiding projective transformations, is as follows:

(1) For each point $\boldsymbol{x}$ in the Schlegel complex $\mathcal{C}(\partial P)\backslash\{F_0\}$ let $\bar{\boldsymbol{x}}$ be the intersection point of H and the segment $[\boldsymbol{x}_0, \boldsymbol{x}]$, and let $\boldsymbol{x}'$ be the point on the segment $[\boldsymbol{x}_0, \boldsymbol{x}]$ such that $[\boldsymbol{x}_0, \bar{\boldsymbol{x}}; \boldsymbol{x}', \boldsymbol{x}]$ form a harmonic quadruple (cross ratio -1).
That is, if $\vec{\boldsymbol{v}}$ is a direction vector such that $\boldsymbol{x} = \boldsymbol{x}_0 + t\vec{\boldsymbol{v}}$ for some $t > 1$ denotes the difference $\boldsymbol{x} - \boldsymbol{x}_0$, while $\bar{\boldsymbol{x}} = \boldsymbol{x}_0 + \vec{\boldsymbol{v}}$ lies on H, then $\boldsymbol{x}' = \boldsymbol{x}_0 + \frac{t}{2t-1}\vec{\boldsymbol{v}}$.

(2) For each face G of the Schlegel complex, $G' := \{\boldsymbol{x}' : \boldsymbol{x} \in G\}$ is the “projectively reflected” copy of G on the other side of H.

(3) The Schlegel cap $\textsc{SchlegelCap}(P, F_0)$ is the regular polytopal ball with faces G, G' and $\operatorname{conv}(G \cup G')$ for faces G in the Schlegel complex.

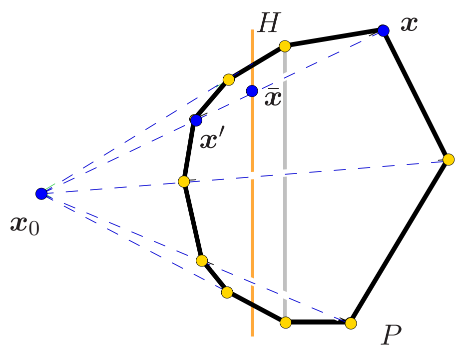


FIGURE 14: Constructing the Schlegel cap via cross ratios.

5 A small cubical 4-polytope with a dual Klein bottle

In this section we present the first instance of a cubical 4-polytope with a non-orientable dual manifold. By Proposition 2.1 this instance is not edge-orientable. Hence, its existence also confirms the conjecture of Hetyei [17, Conj. 2, p. 325]. Apparently this is the first example of a cubical polytope with a non-orientable dual manifold.

Theorem 5.1. *There is a cubical* 4*-polytope* P_{72} *with* f*-vector*

$$f(P_{72}) \quad = \quad (72,\, 196,\, 186,\, 62),$$

one of whose dual manifolds is an immersed Klein bottle of f*-vector* (80, 160, 80)*.*

Step 1. We start with a cubical octahedron O_8, the smallest cubical 3-polytope that is not a cube, with f-vector

$$f(O_8) \quad = \quad (10,\ 16,\ 8).$$

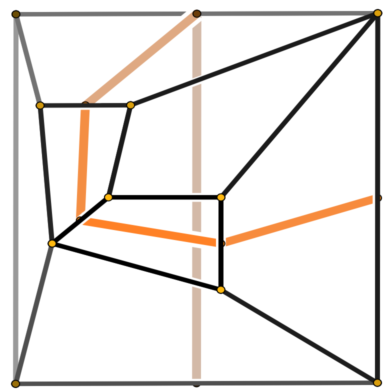


FIGURE 15: The cubical octahedron O_8 positioned in $\mathbb{R}^3$ with a regular square base facet Q and acute dihedral angles at this square base. A part of one dual manifold is highlighted.

We may assume that O_8 is already positioned in $\mathbb{R}^3$ with a regular square base facet Q and acute dihedral angles at this square base; compare the figure below. The f-vector of any Schlegel diagram of O_8 is

$$f(\textsc{Schlegel}(O_8, Q)) \quad = \quad (10,\ 16,\ 7).$$

Let O_8' be a congruent copy of O_8, obtained by reflection of O_8 in its square base followed by a 90° rotation around the axis orthogonal to the base; compare the figure below. This results in a regular 3-ball with cubical 2-skeleton. Its f-vector is

$$f(\mathcal{B}_2) \quad = \quad (16,\ 28,\ 15,\ 2).$$

The special feature of this complex is that it contains a cubical Möbius strip with parallel inner edges of length 9 in its 2-skeleton, as is illustrated in the figure.

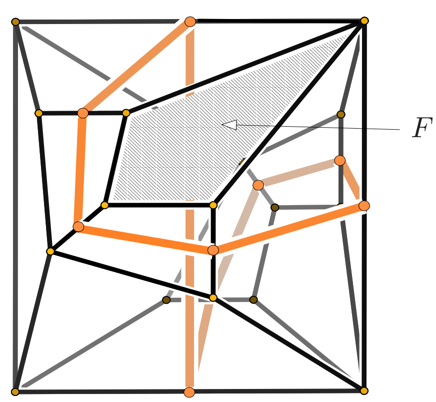


FIGURE 16: The outcome of step 1 of the construction: The 2-cubical convex 3-ball $\mathcal{B}_2$ which contains a Möbius strip with parallel inner edges in the 2-skeleton.

Step 2. Now we perform a Schlegel cap construction on O_8, based on the (unique) facet F of O_8 that is not contained in the Möbius strip mentioned above, and that is not adjacent to the square glueing facet Q. This Schlegel cap has the f-vector

$$f(\mathcal{S}_7) \quad = \quad (20,\ 42,\ 30,\ 7),$$

while its boundary has the f-vector

$$f(\partial\mathcal{S}_7) \quad = \quad (20,\ 36,\ 18).$$

Step 3. The same Schlegel cap operation may be performed on the second copy O_8'. Joining the two copies of the Schlegel cap results in a regular cubical 3-ball $\mathcal{B}_{14}$ with f-vector

$$f(\mathcal{B}_{14}) \quad = \quad (36,\ 80,\ 59,\ 14)$$

whose boundary has the f-vector

$$f(\partial\mathcal{B}_{14}) \quad = \quad (36,\ 68,\ 34).$$

The ball $\mathcal{B}_{14}$ again contains the cubical Möbius strip with parallel inner edges of length 9 as an embedded subcomplex in its 2-skeleton. Compare Figure 17.

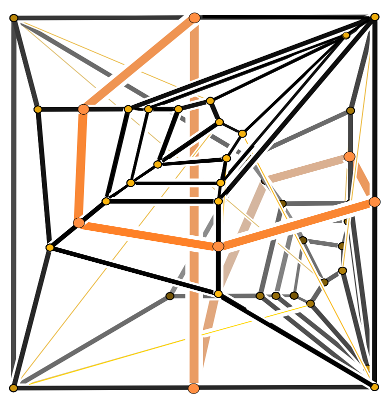


FIGURE 17: The outcome of step 2 of the construction: The cubical convex 3-ball $\mathcal{B}_{14}$ which contains a Möbius strip with parallel inner edges in the 2-skeleton.

Step 4. Now we build the prism over this regular cubical ball, resulting in a regular cubical 4-ball $\mathcal{B}$ whose f-vector is

$$f(\mathcal{B}) \quad = \quad (72,\ 196,\ 198,\ 87,\ 14)$$

and whose support is a cubical 4-polytope $P_{72} := |\mathcal{B}|$ with two copies of the cubical Möbius strip in its 2-skeleton. Its f-vector is

$$f(P_{72}) \quad = \quad (72,\ 196,\ 186,\ 62).$$

A further (computer-supported) analysis of the dual manifolds shows that there are six dual manifolds in total: one Klein bottle of f-vector (80, 160, 80), and five 2-spheres (four with f-vector (20, 36, 18), one with f-vector (36, 68, 34)). All the spheres are embedded, while the Klein bottle is immersed with five double-intersection curves (embedded 1-spheres), but with no triple points. □

6 Constructing cubifications

A lot of construction techniques for cubifications (see Section 2.2) are available in the CW category. In particular, every cubical CW $(d-1)$-sphere $\mathcal{S}^{d-1}$ with an even number of facets admits a CW cubification, that is, a cubical CW d-ball with boundary $\mathcal{S}^{d-1}$, according to Thurston [31], Mitchell [23], and Eppstein [10].

6.1 The Hexhoop template

Yamakawa & Shimada [32] have introduced an interesting polytopal construction in dimension 3 called the *Hexhoop template*; see Figure 18.

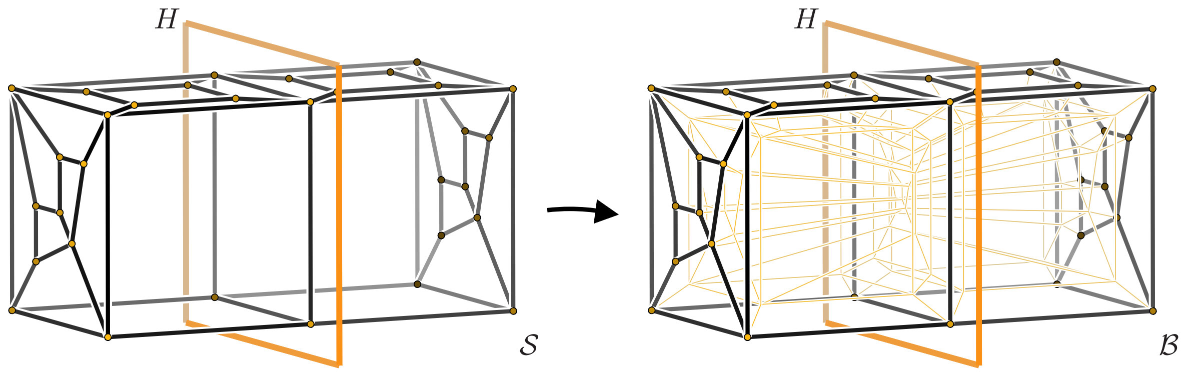


FIGURE 18: The Hexhoop template of Yamakawa & Shimada [32].

Their construction takes as input a 3-polytope P that is affinely isomorphic to a regular 3-cube, a hyperplane H and a cubical subdivision $\mathcal{S}$ of the boundary complex of P such that $\mathcal{S}$ is symmetric with respect to H and H intersects no facet of $\mathcal{S}$ in its relative interior. For such a cubical PL 2-sphere $\mathcal{S}$ the Hexhoop template produces a cubification. A 2-dimensional version is shown in Figure 19.

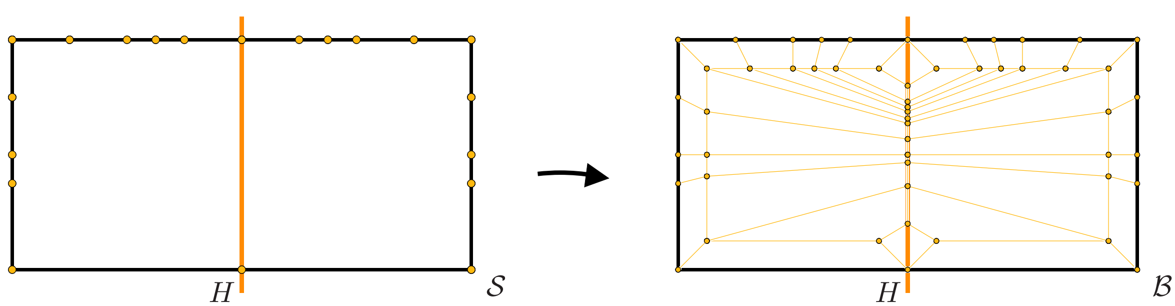


FIGURE 19: A two-dimensional version of the Hexhoop template.

6.2 The generalized regular Hexhoop — overview

In the following we present a *generalized regular Hexhoop* construction. It is a generalization of the Hexhoop template in several directions: Our approach admits arbitrary geometries, works in any dimension, and yields regular cubifications with "prescribed heights on the boundary" (with a symmetry requirement and with the requirement that the intersection of the symmetry

hyperplane and the boundary subdivision is a subcomplex of the boundary subdivision). Figure 20 displays a 2-dimensional cubification (of a boundary subdivision $\mathcal{S}$ of a 2-polytope such that $\mathcal{S}$ is symmetric with respect to a hyperplane H) obtained by our construction.

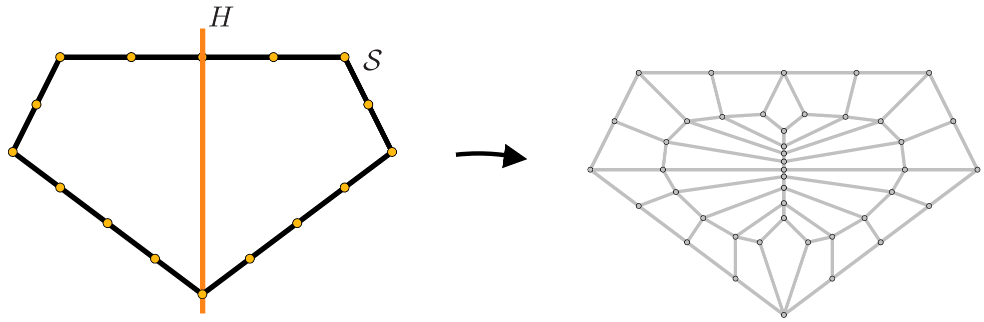


FIGURE 20: A cubification of a boundary subdivision of a pentagon, produced by our *generalized regular Hexhoop* construction.

Not only do we get a cubification, but we may also derive a symmetric lifting function for the cubification that may be quite arbitrarily prescribed on the boundary. The input of our construction is a lifted cubical boundary subdivision $(\mathcal{S}^{d-1}, \psi)$ of a d-polytope P, such that both P and $(\mathcal{S}^{d-1}, \psi)$ are symmetric with respect to a hyperplane H.

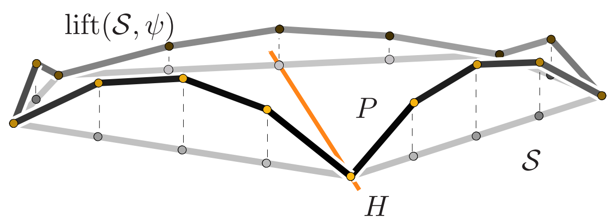


FIGURE 21: An input for the generalized regular Hexhoop construction.

Our approach goes roughly as follows.

(1) We first produce a $(d+1)$-polytope T that is a *symmetric tent* (defined in Section 6.3) over the given lifted boundary subdivision $(\mathcal{S}, \psi)$ of the input d-polytope P. Such a tent is the convex hull of all 'lifted vertices' $(\boldsymbol{v}, \psi(\boldsymbol{v})) \in \mathbb{R}^{d+1}$, $\boldsymbol{v} \in \operatorname{vert}(\mathcal{S})$, and of two *apex points* $\boldsymbol{p}_L, \boldsymbol{p}_R$; compare Figure 22.

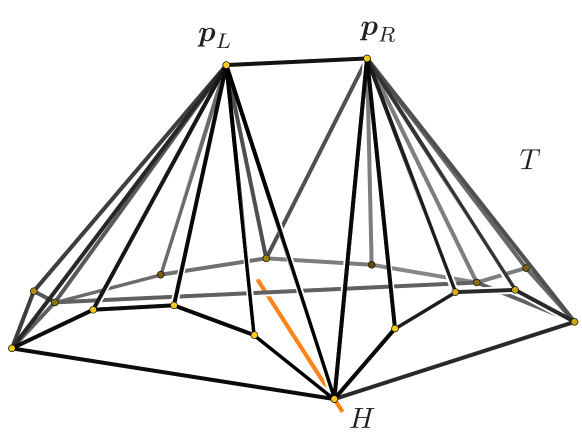


FIGURE 22: A symmetric tent over the lifted boundary subdivision $(\mathcal{S}, \psi)$ of the input d-polytope P.

(2) Truncate T by a hyperplane H' parallel to $\text{aff}(P) = \mathbb{R}^d \subset \mathbb{R}^d \times \{0\}$ that separates the lifted points from the apex points, and remove the upper part.

(3) Add the polytope $R := \text{cone}(\boldsymbol{p}_L, Q) \cap \text{cone}(\boldsymbol{p}_R, Q) \cap H'_+$, where H'_+ is the halfspace with respect to H that contains $\boldsymbol{p}_L$ and $\boldsymbol{p}_R$. Compare Figure 23.

(4) Project the upper boundary complex of the resulting polytope to $\mathbb{R}^d$.

The figures in this section illustrate the generalized regular Hexhoop construction for the 2-dimensional input polytope of Figure 21; the generalized Hexhoop construction for $d = 2$ yields 2-dimensional complexes in $\mathbb{R}^3$. The extension to higher dimensions is immediate, and the case $d = 3$ is crucial for us (see Section 7). It is, however, also harder to visualize: A 3-dimensional generalized regular Hexhoop cubification is shown in Figure 29.

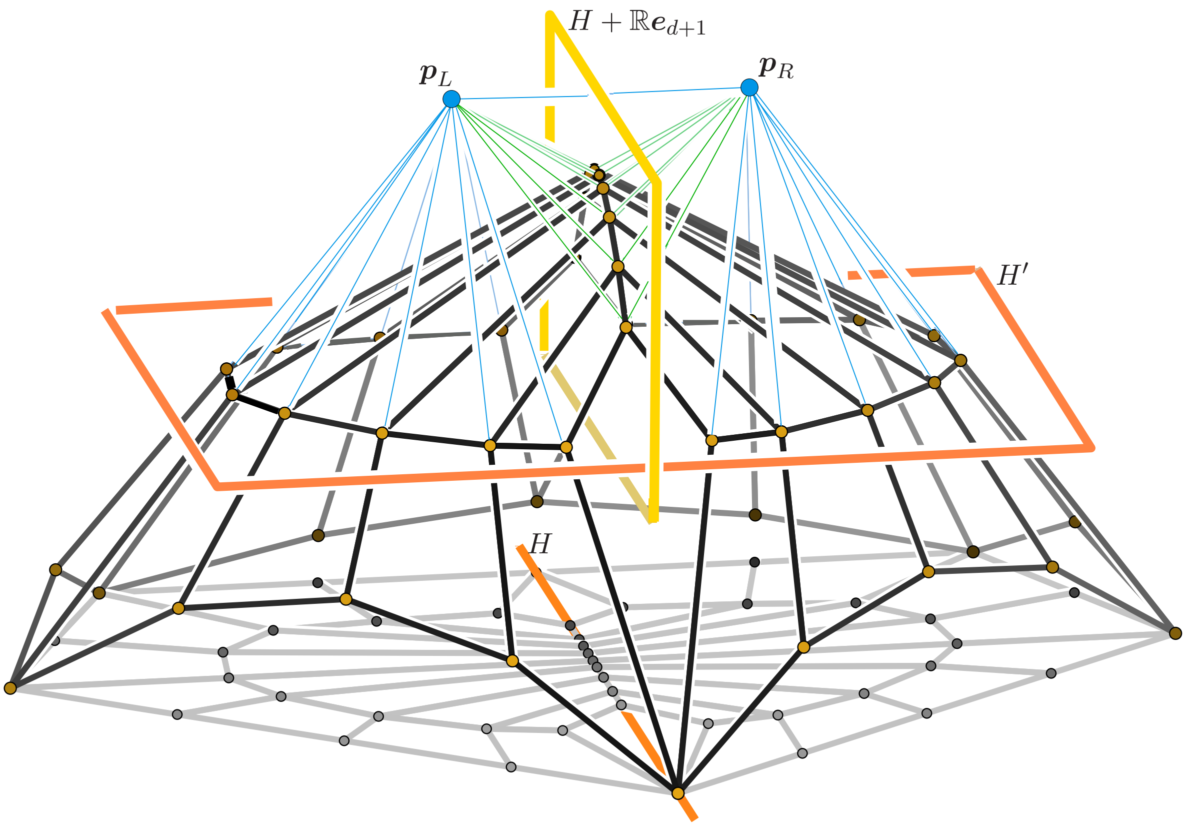


FIGURE 23: Sketch of the generalized regular Hexhoop construction.

6.3 Symmetric tent over a lifted boundary subdivision

Let P be a d-polytope that is symmetric with respect to a hyperplane H in $\mathbb{R}^d$. Choose a positive halfspace H_+ with respect to H. Let $(\mathcal{S}, \psi)$ be a lifted boundary subdivision of P such that $\mathcal{S} \cap H$ is a subcomplex of $\mathcal{S}$. Define $\widetilde{H} := H + \mathbb{R}\boldsymbol{e}_{d+1}$, which is a symmetry hyperplane for $P \subset \mathbb{R}^{d+1}$. The positive halfspace of $\widetilde{H} \subset \mathbb{R}^{d+1}$ will be denoted by $\widetilde{H}_+$.

The *symmetric tent* over $(\mathcal{S}, \psi)$ is the lifted polytopal subdivision $(\mathcal{T}, \phi)$ of P given by the upper faces of the polytope

$$T \ := \ \text{conv}(P \cup \{\boldsymbol{p}_L, \boldsymbol{p}_R\})$$

if $\boldsymbol{p}_L, \boldsymbol{p}_R \in \mathbb{R}^{d+1}$ are two apex points in $\mathbb{R}^{d+1}$ that are symmetric with respect to the hyperplane $\widetilde{H}$, and the upper facets of T are

- pyramids with apex point $\boldsymbol{p}_L$ over facets F of lift$(\mathcal{S},\psi)$ such that $\pi(F) \subset H_+$,
- pyramids with apex point $\boldsymbol{p}_R$ over facets F of lift$(\mathcal{S},\psi)$ such that $\pi(F) \subset H_-$, and
- 2-fold pyramids with apex points $\boldsymbol{p}_L, \boldsymbol{p}_R$ over ridges R of lift$(\mathcal{S},\psi)$ with $\pi(R) \subset H$.

(This requires that $\boldsymbol{p}_L \notin \operatorname{aff}(P)$ and $\pi(\boldsymbol{p}_L) \in \operatorname{relint}(P \cap H_+)$.)

Lemma 6.1. *Assume we are given the following input.*

P	*a convex d-polytope in $\mathbb{R}^d$,*
$(\mathcal{S},\psi)$	*a lifted boundary subdivision of P,*
H	*a hyperplane in $\mathbb{R}^d$ such that* • *P and $(\mathcal{S},\psi)$ are both symmetric with respect to H, and* • *$\mathcal{S} \cap H$ is a subcomplex of $\mathcal{S}$, and*
$\boldsymbol{q}_L, \boldsymbol{q}_R$	*two points in $P \subset \mathbb{R}^d$ such that* • *$\boldsymbol{q}_L \in \operatorname{relint}(P \cap H_+)$, and* • *$\boldsymbol{q}_L, \boldsymbol{q}_R$ are symmetric with respect to $\widetilde{H}$.*

Then for every sufficiently large height $h > 0$ the $(d+1)$-polytope $T := \operatorname{conv}\{\operatorname{lift}(\mathcal{S},\psi), \boldsymbol{p}_L, \boldsymbol{p}_R\}$ with $\boldsymbol{p}_L := (\boldsymbol{q}_L, h) \in \widetilde{H}_+$ and $\boldsymbol{p}_R := (\boldsymbol{q}_R, h) \notin \widetilde{H}_+$ is a symmetric tent over $(\mathcal{S},\psi)$.

This can be shown for instance by using the Patching Lemma (Lemma 3.3).

6.4 The generalized regular Hexhoop in detail

In this section we specify our generalization of the Hexhoop template and prove the following existence statement for cubifications.

Theorem 6.2. *Assume we are given the following input.*

P	*a convex d-polytope in $\mathbb{R}^d$,*
$(\mathcal{S}^{d-1},\psi)$	*a lifted cubical boundary subdivision of P, and*
H	*a hyperplane in $\mathbb{R}^d$ such that* • *P and $(\mathcal{S}^{d-1},\psi)$ are symmetric with respect to H, and* • *$\mathcal{S}^{d-1} \cap H$ is a subcomplex of $\mathcal{S}^{d-1}$.*

Then there is a lifted cubification $(\mathcal{B}^d,\phi)$ of $(\mathcal{S}^{d-1},\psi)$.

The proof relies on the following construction.

Construction 5: Generalized regular Hexhoop

Input:

P	a convex d-polytope P in $\mathbb{R}^d$.
$(\mathcal{S}^{d-1},\psi)$	a lifted cubical boundary subdivision of P.
H	a hyperplane in $\mathbb{R}^d$ such that • P and $(\mathcal{S}^{d-1},\psi)$ are symmetric with respect to H, and • $\mathcal{S}^{d-1} \cap H$ is a subcomplex of $\mathcal{S}^{d-1}$.

Output:

$(\mathcal{B}^d,\phi)$	a symmetric lifted cubification of $(\mathcal{S}^{d-1},\psi)$ given by a cubical d-ball $\mathcal{C}'$ in $\mathbb{R}^{d+1}$.

(1) Choose a positive halfspace H_+ with respect to H, and a point $\boldsymbol{q}_L \in \operatorname{relint}(P \cap H_+)$. Define $\boldsymbol{q}_R := \boldsymbol{p}_L^M$, where the upper index ^{M} denotes the mirrored copy with respect to $\widetilde{H} = H + \mathbb{R}\boldsymbol{e}_{d+1}$.

By Lemma 6.1 there is a height $h > 0$ such that

$$T := \operatorname{conv}\{\operatorname{lift}(\mathcal{S}^{d-1}, \psi), \boldsymbol{p}_L, \boldsymbol{p}_R\}$$

with $\boldsymbol{p}_L := (\boldsymbol{q}_L, h)$ and $\boldsymbol{p}_R := (\boldsymbol{q}_R, h)$ forms a symmetric tent over $(\mathcal{S}^{d-1}, \psi)$.

(2) Choose a hyperplane H' parallel to $\operatorname{aff}(P) \subset \mathbb{R}^d$ that separates $\{\boldsymbol{p}_L, \boldsymbol{p}_R\}$ and $\operatorname{lift}(\mathcal{S}^{d-1}, \psi)$. Let H'_+ be the halfspace with respect to H' that contains $\boldsymbol{p}_L$ and $\boldsymbol{p}_R$.

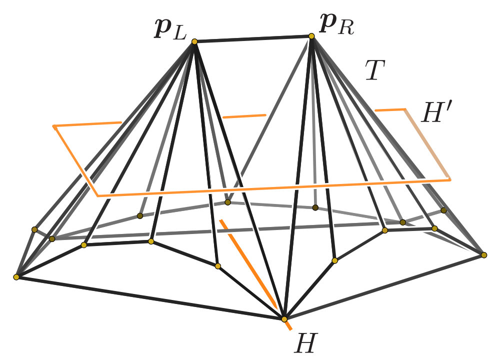


FIGURE 24: Step 2. The hyperplane H' separates $\{\boldsymbol{p}_L, \boldsymbol{p}_R\}$ from $\operatorname{lift}(\mathcal{S}^{d-1}, \psi)$.

(3) Define the "lower half" of the tent T as

$$T_- \;:=\; T \cap H'_-,$$

whose "top facet" is the convex d-polytope $Q := T \cap H'$.

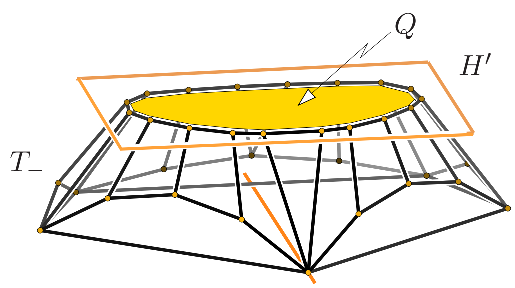


FIGURE 25: Step 3. The "lower half" T_- of T.

(4) Define the two d-polytopes

$$\begin{aligned} Q_L &:= \operatorname{conv}\{\boldsymbol{v} \in \operatorname{vert}(Q) : \boldsymbol{v} \in H_+\}, \\ Q_R &:= \operatorname{conv}\{\boldsymbol{v} \in \operatorname{vert}(Q) : \boldsymbol{v} \in H_-\}. \end{aligned}$$

Let $F_L := H' \cap \operatorname{conv}(\boldsymbol{p}_L, P \cap H)$, the unique facet of Q_L that is not a facet of Q.

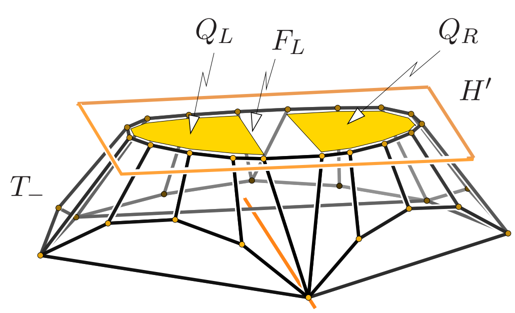


FIGURE 26: Step 4. Define Q_L and Q_R.

(5) Construct the polytope

$$R \;:=\; \mathrm{cone}(\boldsymbol{p}_L, Q) \;\cap\; \mathrm{cone}(\boldsymbol{p}_R, Q) \;\cap\; H'_+.$$

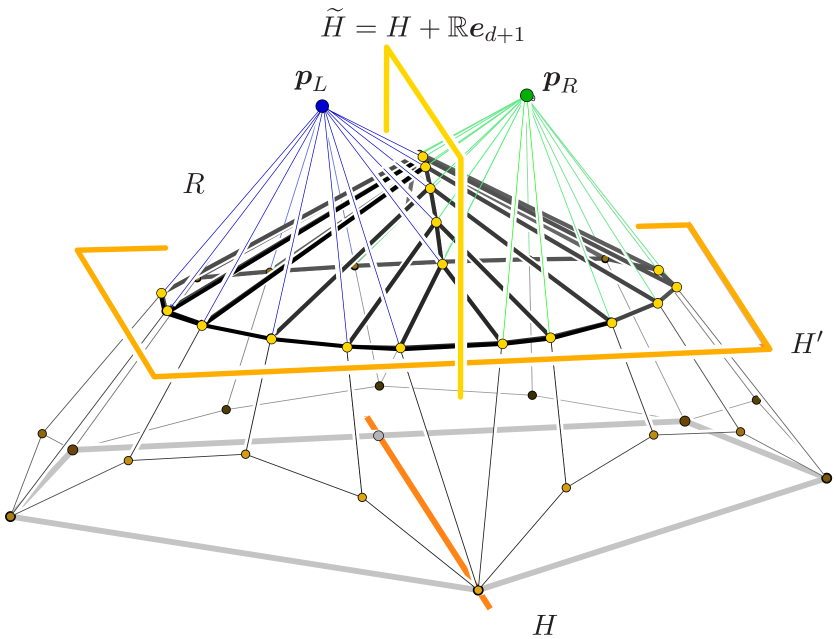


FIGURE 27: Step 5. The polytope $R := \mathrm{cone}(\boldsymbol{p}_L, Q) \cap \mathrm{cone}(\boldsymbol{p}_R, Q) \cap H'_+$.

The complex $\mathcal{C}'$ in question is given by the upper facets of the $(d+1)$-polytope

$$U \;:=\; T_- \cup R.$$

Lemma 6.3 (Combinatorial structure of Q).
The vertex set of Q consists of

- *the points* $\mathrm{conv}(\boldsymbol{p}_L, \boldsymbol{v}) \cap H'$ *for vertices* $\boldsymbol{v} \in \mathrm{vert}(\mathrm{lift}(\mathcal{S}, \psi))$ *such that* $\pi(\boldsymbol{v}) \subset H_+$*, and*
- *the points* $\mathrm{conv}(\boldsymbol{p}_R, \boldsymbol{v}) \cap H'$ *for vertices* $\boldsymbol{v} \in \mathrm{vert}(\mathrm{lift}(\mathcal{S}, \psi))$ *such that* $\pi(\boldsymbol{v}) \subset H_-$*.*

The facets of Q are

(a) *the combinatorial cubes* $\mathrm{conv}(\boldsymbol{p}_L, F) \cap H'$ *for facets F of* $\mathrm{lift}(\mathcal{S}, \psi)$ *such that* $F \subset \widetilde{H}_+$*,*
(b) *the combinatorial cubes* $\mathrm{conv}(\boldsymbol{p}_R, F) \cap H'$ *for facets F of* $\mathrm{lift}(\mathcal{S}, \psi)$ *such that* $F \subset \widetilde{H}_-$*,*
(c) *the combinatorial cubes* $\mathrm{conv}(\boldsymbol{p}_L, \boldsymbol{p}_R, F) \cap H'$ *for $(d-2)$-faces F of* $\mathrm{lift}(\mathcal{S}, \psi)$ *with* $F \subset \widetilde{H}$*.*

Proof. By the definition of a symmetric tent, upper facets of the symmetric tent T are

- the pyramids with apex point $\boldsymbol{p}_L$ over facets F of $\mathrm{lift}(\mathcal{S}, \psi)$ such that $F \subset \widetilde{H}_+$,
- the pyramids with apex point $\boldsymbol{p}_R$ over facets F of $\mathrm{lift}(\mathcal{S}, \psi)$ such that $F \subset \widetilde{H}_-$, and
- the 2-fold pyramids with apex points $\boldsymbol{p}_L, \boldsymbol{p}_R$ over ridges R of $\mathrm{lift}(\mathcal{S}, \psi)$ with $R \subset \widetilde{H}$.

Since Q is the intersection of T with H, the polytope Q has the vertices and facets listed above. It remains to show that the facets of type (c) are combinatorial cubes. Let F be a $(d-2)$-face of $\mathrm{lift}(\mathcal{S}, \psi)$ such that $F \subset \widetilde{H}$. Every point on the facet lies in the convex hull of F with a unique point on the segment $[\boldsymbol{p}_L, \boldsymbol{p}_R]$. Thus the facet is combinatorially isomorphic to a prism over F. $\square$

Let a *d-dimensional half-cube* be the product of a combinatorial $(d-2)$-cube and a triangle. A *combinatorial half-cube* is a polytope combinatorially isomorphic to a half-cube.

Lemma 6.4 (Combinatorial structure of T_-).
The vertices of T_- are the vertices of $\mathrm{lift}(\mathcal{S}, \psi)$ *and the vertices of Q. Furthermore, the upper facets of T_- are*

(a) *the combinatorial cubes* $\mathrm{cone}(\boldsymbol{p}_L, F) \cap H'_- \cap (\mathbb{R}^d \times \mathbb{R}_+)$ *for facets F of Q such that $F \subset \widetilde{H}_+$,*
(b) *the combinatorial cubes* $\mathrm{cone}(\boldsymbol{p}_R, F) \cap H'_- \cap (\mathbb{R}^d \times \mathbb{R}_+)$ *for facets F of Q such that $F \subset \widetilde{H}_-$,*
(c) *the combinatorial half-cubes* $\mathrm{cone}(\boldsymbol{p}_L, F) \cap \mathrm{cone}(\boldsymbol{p}_R, F) \cap H'_-$ *for facets R of Q that intersect $\widetilde{H}$, and*
(d) *Q.*

The facet defining hyperplanes of the upper facets of T_- are

(a) $\mathrm{aff}(\boldsymbol{p}_L, F)$ *for facets F of Q such that $F \subset \widetilde{H}_+$,*
(b) $\mathrm{aff}(\boldsymbol{p}_R, F)$ *for facets F of Q such that $F \subset \widetilde{H}_-$,*
(c) $\mathrm{aff}(\boldsymbol{p}_L, \boldsymbol{p}_R, F)$ *for facets F of Q that intersect $\widetilde{H}$, and*
(d) $\mathrm{aff}(Q)$.

Proof. Since T_- is the intersection of T with H'_-, the upper facets of T_- are given by Q plus the intersections of the upper facets of T with H'_-, and the vertices of T_- are the vertices of T and the vertices of Q. ∎

Lemma 6.5 (Combinatorial structure of R).
The set of vertices of R consists of the vertices of Q and all points in $V'' := \mathrm{vert}(R) \setminus \mathrm{vert}(Q)$. Furthermore, the set of (all) facets of R consists of

(a) *the combinatorial cubes* $\mathrm{conv}(\boldsymbol{p}_R, F) \cap \widetilde{H}_+$ *for facets F of Q such that $F \subset \widetilde{H}_+$,*
(b) *the combinatorial cubes* $\mathrm{conv}(\boldsymbol{p}_L, F) \cap \widetilde{H}_+$ *for facets F of Q such that $F \subset \widetilde{H}_-$,*
(c) *the combinatorial half-cubes* $\mathrm{conv}(\boldsymbol{p}_R, F) \cap \mathrm{conv}(\boldsymbol{p}_L, F)$ *for facets F of Q that intersect $\widetilde{H}$, and*
(d) *Q.*

The set of facet defining hyperplanes of the facets of R consists of

(a) $\mathrm{aff}(\boldsymbol{p}_R, F)$ *for facets F of Q such that $F \subset \widetilde{H}_+$,*
(b) $\mathrm{aff}(\boldsymbol{p}_L, F)$ *for facets F of Q such that $F \subset \widetilde{H}_-$,*
(c) $\mathrm{aff}(\boldsymbol{p}_L, \boldsymbol{p}_R, F)$ *for facets F of Q such that F intersects $\widetilde{H}$, and*
(d) $\mathrm{aff}(Q)$.

Proof of Theorem 6.2. We show that the complex $\mathcal{C}'$ given by the upper facets of the polytope U of Construction 5 determines a lifted cubification $(\mathcal{B}^d, \phi)$ of $(\mathcal{S}^{d-1}, \psi)$.

First observe that no vertex of T_- is beyond a facet of R, and no vertex of R is beyond a facet of T_-. Hence the boundary of $U = \mathrm{conv}(T_- \cup R)$ is the union of the two boundaries of the two polytopes, excluding the relative interior of Q.
Define the vertex sets $V := \mathrm{vert}(\mathrm{lift}(\mathcal{S}t, \psi))$, $V' := \mathrm{vert}(Q)$ and $V'' := \mathrm{vert}(R) \setminus V'$. Then

- each vertex of V is beneath each facet of R that is of type (a) or (b), and
- each vertex of V'' is beneath each facet of T_- that is of type (a) or (b).

Hence these four types of facets are facets of U that are combinatorial cubes, and the set of vertices of U is given by the union of V, V' and V''. It remains to show that each hyperplane $\mathrm{aff}(\boldsymbol{p}_L, \boldsymbol{p}_R, F)$, where F is a facet of Q that intersects $\widetilde{H}$, is the affine hull of a cubical facet

of U. To see this, observe that there are two facets F_+, F_- of R, T_- respectively, that are both contained in the affine hull of F. These two facets F_+, F_- are both half-cubes that intersect in a common $(d-1)$-cube, namely F. Furthermore, all vertices of F_+ and of F_- that are not contained in $\operatorname{aff}(F)$ are contained in $\widetilde{H}$. Hence the union of F_+ and F_- is a combinatorial cube.

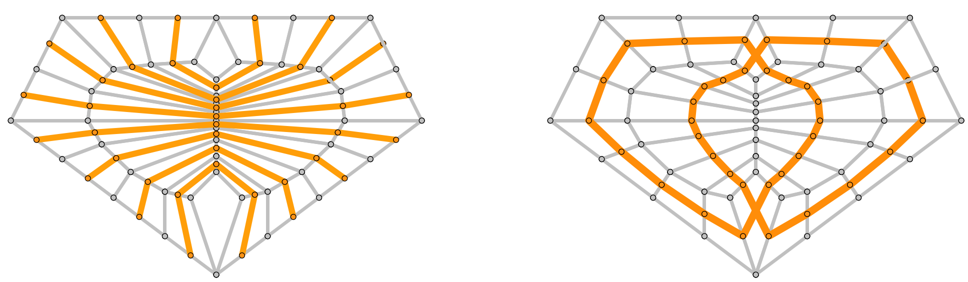
Thus every upper facet of U is a combinatorial cube. Furthermore, $\pi(R) = \pi(Q)$ and $\pi(T_-) = |P|$, so the upper facets of U determine a lifted cubical subdivision of $(\mathcal{S}^{d-1}, \psi)$. □

Proposition 6.6 (Dual manifolds). *Up to PL-homeomorphism, the generalized regular Hexhoop cubification $\mathcal{B}^d$ of $\mathcal{S}^{d-1}$ has the following dual manifolds:*

- *$\mathcal{N} \times I$ for each dual manifold $\mathcal{N}$ (with or without boundary) of $\mathcal{S}_L = \mathcal{S}^{d-1} \cap \widetilde{H}_+$,*
- *two $(d-1)$-spheres "around" Q, Q^M, respectively, where the upper index ^{M} denotes the mirrored copy.*

Proof. The "main part" of the complex $\mathcal{B}^d$ may be viewed as a prism of height 4, whose dual manifolds are of the form $\mathcal{N} \times I$, as well as four $(d-1)$-balls. This prism is then modified by glueing a full torus (product of the $(d-2)$-sphere $\mathcal{S}^{d-1} \cap \mathcal{H}$ with a square I^2) into its "waist." This extends the dual manifolds $\mathcal{N} \times I$ without changing the PL-homeomorphism type, while closing the four $(d-1)$-balls into two intersecting, embedded spheres.

We refer to Figure 28 (case $d = 2$) and Figure 29 ($d = 3$) for geometric intuition.



(a) Dual product manifolds $\mathcal{N} \times I$ (b) Dual spheres

FIGURE 28: The dual manifolds of a 2-dimensional generalized regular Hexhoop.

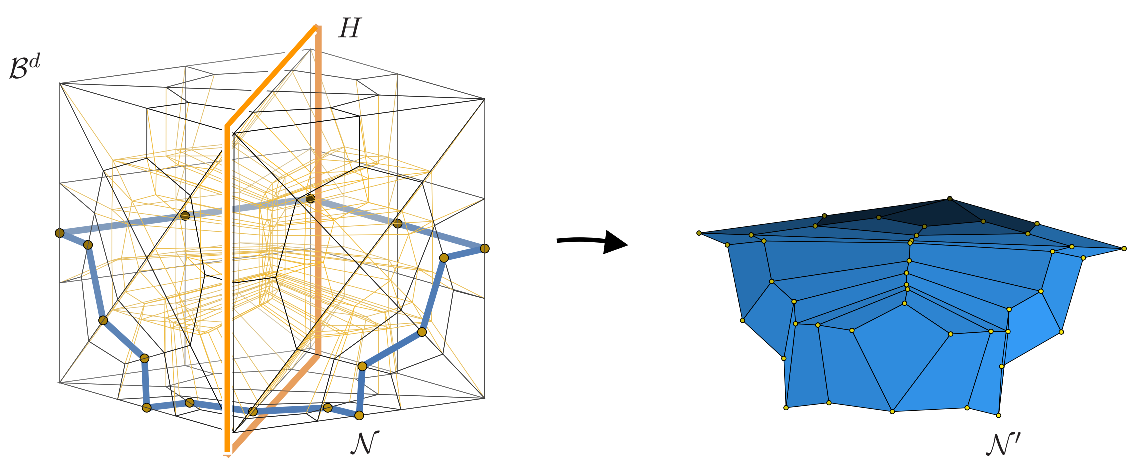


FIGURE 29: A 3-dimensional cubification produced by the generalized regular Hexhoop construction. For every embedded dual circle $\mathcal{N}$ which intersects $H_+ \setminus H$ and $H_- \setminus H$, there is an embedded dual 2-ball $\mathcal{N}'$ with boundary $\mathcal{N}$ in the the cubification. (This is a cubification for the case "single5" introduced in Section 7.)

□

7 Cubical 4-polytopes with prescribed dual manifold immersions

Now we use our arsenal of cubical construction techniques for the construction of cubifications with prescribed dual 2-manifold immersions, and thus approach our main theorem.

For this we ask for our input to be given by normal crossing PL-immersions whose local geometric structure is rather special: We assume that $\mathcal{M}^{d-1}$ is a $(d-1)$-dimensional cubical PL-manifold, and $j : \mathcal{M}^{d-1} \looparrowright \mathbb{R}^d$ is a *grid immersion*, a cubical normal crossing codimension one immersion into $\mathbb{R}^d$ equipped with the standard unit cube structure.

7.1 From PL immersions to grid immersions

In view of triangulation and approximation methods available in PL and differential topology, the above assumptions are not so restrictive. (See, however, Dolbilin et al. [9] for extra problems and obstructions that may arise without the PL assumption, and if we do not admit subdivisions, even for the high codimension embeddings/immersions.)

Proposition 7.1. *Every locally flat normal crossing immersion of a compact $(d-1)$-manifold into $\mathbb{R}^d$ is PL-equivalent to a grid immersion of a cubification of the manifold into the standard cube subdivision of $\mathbb{R}^d$.*

Proof. may replace any PL-immersion of $\mathcal{M}^{d-1}$ by a simplicial immersion into a suitable triangulation of $\mathbb{R}^d$. The vertices of $j(\mathcal{M}^{d-1})$ may be perturbed into general position.

Now we overlay the polyhedron $j(\mathcal{M}^{d-1})$ with a cube structure of $\mathbb{R}^d$ of edge length ε for suitably small $\varepsilon > 0$, such that the vertices of $j(\mathcal{M}^{d-1})$ are contained in the interiors of distinct d-cubes.

Then working by induction on the skeleton, within each face of the cube structure, the restriction of $j(\mathcal{M}^{d-1})$ to a k-face — which by local flatness consists of one or several $(k-1)$-cells that intersect transversally — is replaced by a standard cubical lattice version that is supposed to run through the interior of the respective cell, staying away distance ε' from the boundary of the cell; here we take different values for ε' in the situation where the immersion is not embedded at the vertex in question, that is, comes from several disjoint neighborhoods in $\mathcal{M}^{d-1}$.

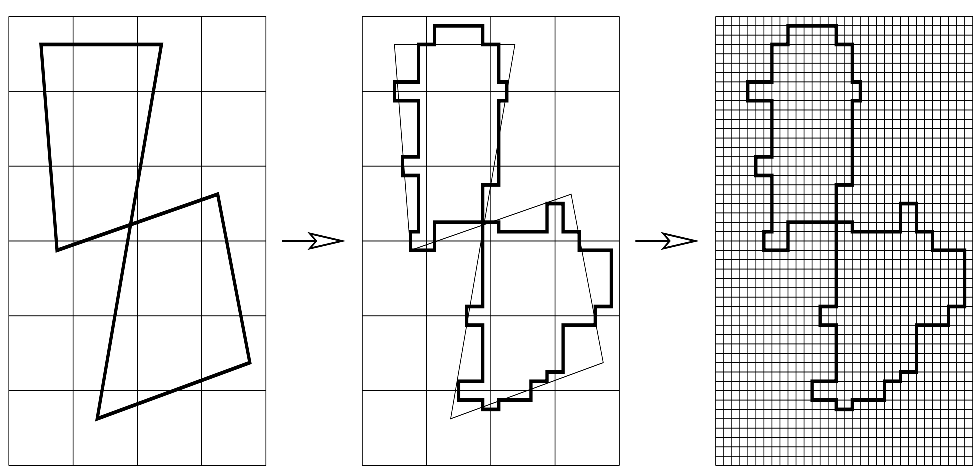


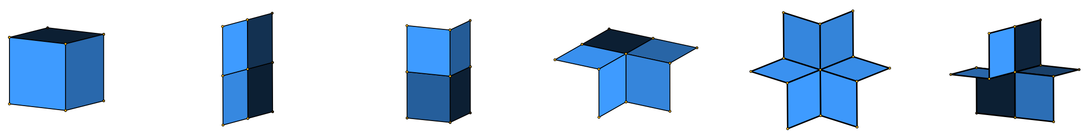
Figure 30: Illustration of the proof of Proposition 7.1.

The resulting modified immersion into $\mathbb{R}^d$ will be cellular with respect to a standard cube subdivision of edge length $\frac{1}{N}\varepsilon$ for a suitable large N. Figure 30 illustrates this for $d = 2$. □

7.2 Vertex stars of grid immersions of surfaces

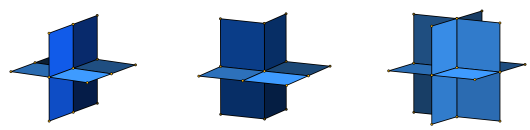
From now on, we restrict our attention to the case of $d = 3$, that is, 2-manifolds and 4-polytopes.

There are nine types of vertex stars of grid immersions of surfaces, namely the following five vertex stars of a regular vertex,



single3 single4a single4b single5 single6a single6b

plus two vertex stars with double intersection and the vertex star of a triple intersection point:

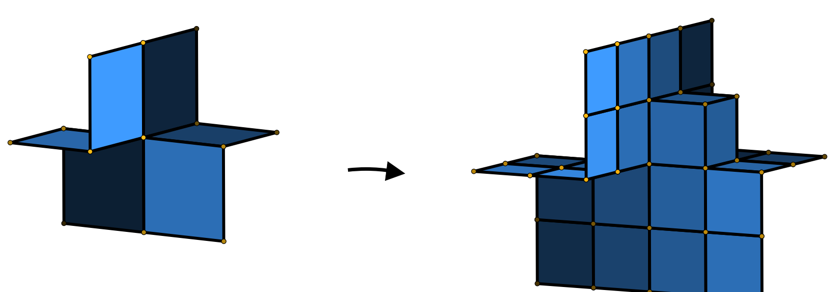


double8a double8b triple12

For the constructions below we will require that the grid immersion $j : \mathcal{M}^2 \looparrowright \mathbb{R}^3$ that we start out with is *locally symmetric*, that is, that at each vertex $\boldsymbol{w}$ of $j(\mathcal{M}^2)$ there is plane H through $\boldsymbol{w}$ such that for each vertex $\boldsymbol{v}$ with $j(\boldsymbol{v}) = \boldsymbol{w}$ the image of the vertex star of $\boldsymbol{v}$ is symmetric with respect to H. Thus we require that H is a symmetry plane separately for each of the (up to three) local sheets that intersect at $\boldsymbol{w}$. Such a plane H is necessarily of the form $x_i = k$, $x_i + x_j = k$, or $x_i - x_j = k$. In the first case we say H is a *coordinate hyperplane*, and in other cases it is *diagonal*.

Proposition 7.2. *Any grid immersion of a compact cubical* 2*-manifold into* $\mathbb{R}^3$ *is equivalent to a locally symmetric immersion of the same type.*

Proof. All the vertex stars displayed above satisfy the local symmetry condition, with a single exception, namely the star “single6b” of a regular vertex with six adjacent quadrangles. As indicated in Figure 31, a local modification of the surface solves the problem (with a suitable refinement of the standard cube subdivision). ☐



single6b

FIGURE 31: Local modification used to “repair” the case “single6a.”

7.3 Main theorem (2-manifolds into cubical 4-polytopes)

Theorem 7.3. *Let $j\colon \mathcal{M} \looparrowright \mathbb{R}^3$ be a locally flat normal crossing immersion of a compact 2-manifold (without boundary) $\mathcal{M}$ into $\mathbb{R}^3$.*

Then there is a cubical 4-polytope P with a dual manifold $\mathcal{M}'$ and associated immersion $y : \mathcal{M}' \looparrowright |\partial P|$ such that the following conditions are satisfied:

(i) *$\mathcal{M}'$ is a cubical subdivision of $\mathcal{M}$, and the immersions j (interpreted as a map to $\mathbb{R}^3 \cup \{\infty\} \cong S^3$) and y are PL-equivalent.*

(ii) *The number of facets of P is congruent modulo 2 to the number $t(j)$ of triple points of the immersion j.*

(iii) *If the given surface $\mathcal{M}$ is non-orientable and of odd genus, then the cubical 4-polytope P has an odd number of facets.*

The core of our proof is the following construction of cubical 3-balls with a prescribed dual manifold immersions.

Construction 6: REGULAR CUBICAL 3-BALL WITH A PRESCRIBED DUAL MANIFOLD

Input: A 2-dimensional closed (that is, compact and without boundary) cubical PL-surface $\mathcal{M}$, and a locally symmetric codimension one grid immersion $j\colon \mathcal{M} \looparrowright |P_3(\ell_1, \ell_2, \ell_3)| \subset \mathbb{R}^3$.

Output: A regular convex 3-ball $\mathcal{B}$ with a dual manifold $\mathcal{M}'$ and associated immersion $y : \mathcal{M}' \looparrowright |\mathcal{B}|$ such that the following conditions are satisfied:

(i) $\mathcal{M}'$ is a cubical subdivision of $\mathcal{M}$, and the immersions j and y are PL-equivalent.

(ii) The number of facets of $\mathcal{B}$ is congruent modulo two to the number $t(j)$ of triple points of the immersion j.

(1) Raw complex. Let $\mathcal{A}$ be a copy of the pile of cubes $P_3(\ell_1+1, \ell_2+1, \ell_3+1)$ with all vertex coordinates shifted by $-\frac{1}{2}\mathbb{1}$. (Hence $x_i \in \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots, \ell_i + \frac{1}{2}\}$ for each vertex $\boldsymbol{x} \in \mathrm{vert}(A)$.)

Due to the local symmetry of the immersion, and the choice of the vertex coordinates of $\mathcal{A}$, the following holds:

- ▷ Each vertex of $j(\mathcal{M})$ is the barycenter of a 3-cube C of $\mathcal{A}$.
- ▷ For each 3-cube C of $\mathcal{A}$ the restriction $(C, j(\mathcal{M}) \cap C)$ is locally symmetric.

(2) Local subdivisions. We construct the lifted cubical subdivision $\mathcal{B}$ of $\mathcal{A}$ by induction over the skeleton: For $k = 1, 2, 3$, $\mathcal{B}^k$ will be a lifted cubical subdivision of the k-skeleton $\mathcal{F}_k(\mathcal{A})$, with the final result $\mathcal{B} := \mathcal{B}^3$. For each k-face $F \in \mathcal{A}$ we take for the restriction $\mathcal{B}^k \cap F$ a congruent copy from a finite list of *templates*.

Consider the following invariants (for $k \in \{1, 2, 3\}$).

(I_k1) *Consistency requirement.*
For every k-face $Q \in \mathcal{F}_k(\mathcal{A})$ and every facet F of Q, the induced subdivision $\mathcal{B}^k \cap F$ equals $\mathcal{B}^{k-1} \cap F$.

(I$_k$2) *PL equivalence requirement.*
For every k-face $Q \in \mathcal{F}_k(\mathcal{A})$ and every dual manifold $\mathcal{N}$ of Q (with boundary) the cubical subdivision $\mathcal{B}^k \cap Q$ has a dual manifold that is PL-equivalent to $j(\mathcal{N}) \cap Q$.

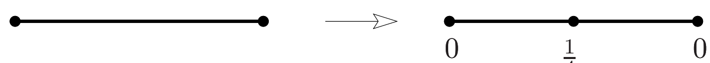
(I$_k$3) *Symmetry requirement.*
Every symmetry of $(Q, j(\mathcal{M}) \cap Q)$ for a k-face $Q \in \mathcal{F}_k(\mathcal{A})$ that is a symmetry of each sheet of $j(\mathcal{M}) \cap Q$ separately is a symmetry of $(Q, \mathcal{B}^k \cap Q)$.

(I$_k$4) *Subcomplex requirement.*
For every diagonal symmetry hyperplane H_Q of a facet Q of $\mathcal{A}$ and every facet F of Q the (lifted) induced subdivision $\mathcal{B}^k \cap (F \cap H)$ is a (lifted) subcomplex of $\mathcal{B}^k$.

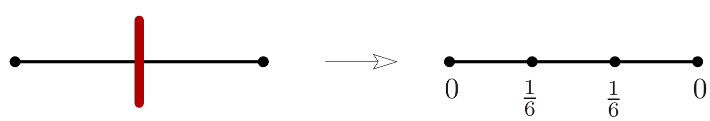
These invariants are maintained while iteratively constructing $\mathcal{B}^1$ and $\mathcal{B}^2$. The resulting lifted cubical subdivision $\mathcal{B}^3$ of $\mathcal{A}$ will satisfy (I$_3$1) and (I$_3$2), but not in general the other two conditions.

(3) Subdivision of edges. Let e be an edge of $\mathcal{A}$.

- If e is not intersected by the immersed manifold, then we subdivide the edge by an affine copy $\mathcal{B}^1_e$ of the following lifted subdivision $\mathcal{U}_2 := (\mathcal{U}'_2, h)$ of $P_1(2)$:

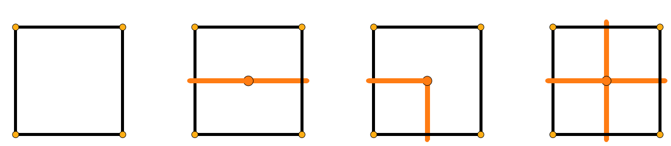


- If e is intersected by the immersed manifold, then we subdivide the edge by an affine copy $\mathcal{B}^1_e$ of the following lifted subdivision $\mathcal{U}_3 := (\mathcal{U}'_3, h)$ of $P_1(3)$:



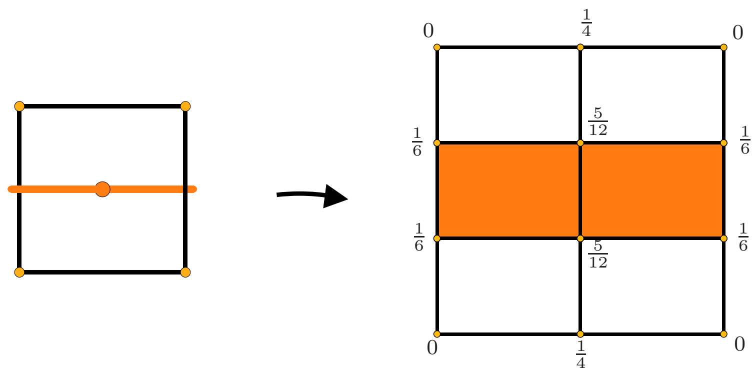
Observe that (I$_1$1)–(I$_1$4) are satisfied.

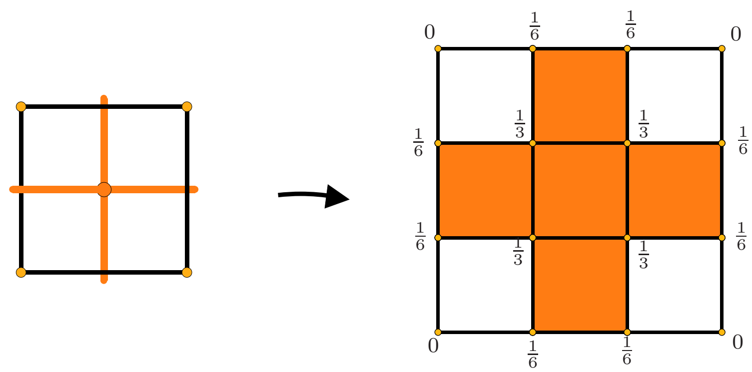
(4) Subdivision of 2-faces. Let Q be a quadrangle of $\mathcal{A}$, and $\boldsymbol{w}$ the unique vertex of $j(\mathcal{M})$ that is contained in Q. There are four possible types of restrictions of the grid immersion to Q:



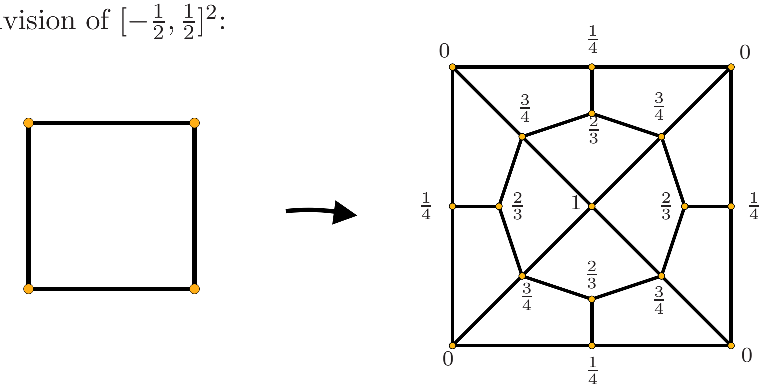
empty   single2a   single2b   double

(a) In the cases "single2a" and "double" there is a coordinate hyperplane H such that $(Q, j(\mathcal{M}) \cap Q)$ is symmetric with respect to H, and a vertex $\boldsymbol{v}$ of $\mathcal{M}$ such that $j(\boldsymbol{v}) = \boldsymbol{w}$ and the image of the vertex star is contained in H. Let F be a facet of Q that does not intersect H. Then $\mathcal{B}^2_Q$ is taken to be a copy of the product $(\mathcal{B}^1 \cap F) \times \mathcal{U}_3$:

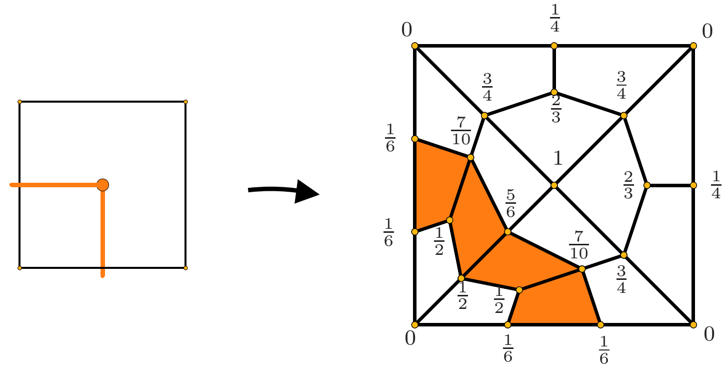




(b) If the immersion does not intersect Q, then $\mathcal{B}_Q^2$ is a copy of the lifted cubical 2-complex $\mathcal{V}$ which arises as the cubical barycentric subdivision of the stellar subdivision of $[-\frac{1}{2}, \frac{1}{2}]^2$:

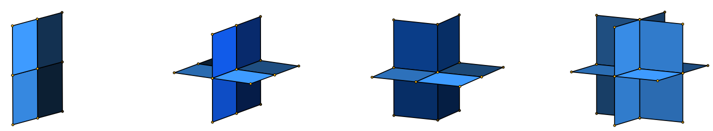


(c) In the case "single2b" we define $\mathcal{B}_Q^2$ as an affine copy of the lifted cubical 2-complex $\mathcal{V}'$, which is given by $\mathcal{V}$ truncated by four additional planes:



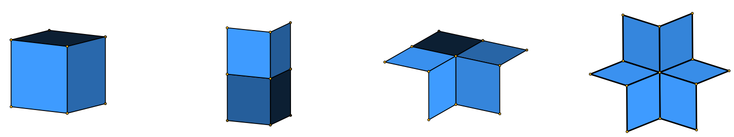
Observe that the conditions ($\mathrm{I}_2 1$)–($\mathrm{I}_2 4$) are satisfied.

(5) **Subdivision of 3-cubes.** Let Q be a facet of $\mathcal{A}$ and $\boldsymbol{w}$ the unique vertex of $j(\mathcal{M})$ that is mapped to the barycenter of Q. Let $\mathcal{S} := \mathcal{B}^2 \cap Q$ be the induced lifted cubical boundary subdivision of Q.
All templates for the lifted cubification $\mathcal{B}^3_Q$ of $\mathcal{S}$ arise either as a generalized regular Hexhoop, or as a product of $\mathcal{U}_3$ with a lifted cubical subdivision of a facet of Q.
(a) For the following four types of vertex stars we use a product with $\mathcal{U}_3$:



single4a double8a double8b triple12

In all these cases there is a coordinate symmetry plane H such that $H \cap Q$ is a sheet of $j(\mathcal{M}) \cap Q$. Hence all facets of Q that intersect H are subdivided by $\mathcal{U}_3 \times \mathcal{U}_3$ or $\mathcal{U}_3 \times \mathcal{U}_2$. Let F be one of the two facets of Q that do not intersect H. Then the product $(\mathcal{B}^2 \cap F) \times \mathcal{U}_3$ yields the lifted subdivision $\mathcal{B}^3_Q$ of Q. Clearly $\mathcal{B}^3_Q$ is consistent with ($\mathrm{I}_3$1) and ($\mathrm{I}_3$2).
(b) In the remaining five cases we take a generalized regular Hexhoop with a diagonal plane of symmetry of Q to produce $\mathcal{B}^3_Q$:



empty single3 single4b single5 single6a

In each of these cases, $(Q, Q \cap j(\mathcal{M}))$ has a diagonal plane H of symmetry. This plane intersects the relative interior of two facets of Q. Since ($\mathrm{I}_2$4) holds, no facet of $\mathcal{S} = \mathcal{B}^2 \cap Q$ intersects H in its relative interior. By ($\mathrm{I}_2$3) the lifted boundary subdivision $\mathcal{S}$ is symmetric with respect to H. Hence all preconditions of the generalized regular Hexhoop construction are satisfied. The resulting cubification $\mathcal{B}^3_Q$ satisfies ($\mathrm{I}_3$1) and ($\mathrm{I}_3$2).

7.4 Correctness

Proposition 7.4. *Let $\mathcal{M}$ be a 2-dimensional closed cubical PL-surface, and $j\colon \mathcal{M} \looparrowright \mathbb{R}^3$ a locally symmetric codimension one grid immersion.*

Then the cubical 3-ball $\mathcal{B}$ given by Construction 6 has the following properties:

(i) *$\mathcal{B}$ is regular, with a lifting function ψ.*

(ii) *There is a dual manifold $\mathcal{M}'$ of $\mathcal{B}$ and associated immersion $y : \mathcal{M}' \looparrowright |\mathcal{B}|$ such that $\mathcal{M}'$ is a cubical subdivision of $\mathcal{M}$, and the immersions j and y are PL-equivalent.*

(iii) *The number of facets of $\mathcal{B}$ is congruent modulo two to the number $t(j)$ of triple points of the immersion j.*

(iv) *There is a lifted cubification $\mathcal{C}$ of $(\partial\mathcal{B}, \psi|_{\partial\mathcal{B}})$ with an even number of facets.*

Proof. (i) *Regularity.* By construction the lifting functions ψ_F, $F \in \mathrm{fac}(\mathcal{A})$, satisfy the consistency precondition of the Patching Lemma (Lemma 3.3). Since every pile of cube is regular the Patching Lemma implies that $\mathcal{B}$ is regular, too.

(ii) *PL-equivalence of manifolds* is guaranteed by Property ($\mathrm{I}_3 2$).

(iii) *Parity of the number of facets.* For each 3-cube Q of $\mathcal{A}$, its cubification $\mathcal{B}^3_Q$ is either a product $\mathcal{B}^2_F \times \mathcal{U}_3$ (where $\mathcal{B}^2_F$ is a cubification of a facet F of Q), or the outcome of a generalized regular Hexhoop construction. In the latter case the the number of facets of $\mathcal{B}^3_Q$ is even. In the first case the number of facets depends on the number of 2-faces of $\mathcal{B}^2_F$. The number of quadrangles of $\mathcal{B}^2_F$ is odd only in the case "double," if $j(\mathcal{M}) \cap F$ has a double intersection point. Hence, $f_3(\mathcal{B}^3_Q)$ is odd if and only if the immersion j has a triple point in Q.

(iv) *Alternative cubification.* Applying Construction 6 to $P_3(\ell_1, \ell_2, \ell_3)$ without an immersed manifold yields a regular cubification $\mathcal{C}$ of $\partial\mathcal{B}$ with the same lifting function as $\mathcal{B}$ on the boundary. Since the immersion $\emptyset \looparrowright \mathbb{R}^3$ has no triple points the number of facets of $\mathcal{C}$ is even. ☐

7.5 Proof of the main theorem

Proof of Theorem 7.3. Let $j\colon \mathcal{M} \looparrowright \mathbb{R}^3$ be a locally flat normal crossing immersion of a compact $(d-1)$-manifold $\mathcal{M}$ into $\mathbb{R}^d$. By Proposition 7.1 and Proposition 7.2 there is a cubical subdivision $\mathcal{M}'$ of $\mathcal{M}$ with a locally symmetric, codimension one grid immersion $j'\colon \mathcal{M}' \looparrowright \mathbb{R}^3$ that is PL-equivalent to j.

Construct a convex cubical 3-ball $\mathcal{B}$ with prescribed dual manifold immersion j' as described above. By Proposition 7.4(i) the ball $\mathcal{B}$ is regular, and by Proposition 7.4(iv) there is a cubification $\mathcal{C}$ of $\partial\mathcal{B}$ with an even number of facets and the same lifting function on the boundary.

The lifted prism over $\mathcal{B}$ and $\mathcal{C}$ (Construction 3 of Section 4.2). This yields a cubical 4-polytope P with

$$f_3(P) = f_3(\mathcal{B}) + f_3(\mathcal{C}) + f_2(\partial\mathcal{B}),$$

whose boundary contains $\mathcal{B}$ and thus has a dual manifold immersion PL-equivalent to j.

For (ii) observe that for every cubical 3-ball the number of facets of the boundary is even. Hence $f_2(\partial\mathcal{B})$ is even. Since the number of facets of $\mathcal{C}$ is even, we obtain

$$f_3(P) \;\equiv\; f_3(\mathcal{B}) \;\equiv\; t(j) \mod 2.$$

Now consider (iii). By a famous theorem of Banchoff [4] the number of triple points of a normal crossing codimension one immersion of a surface has the same parity as the Euler characteristic. Hence, if $\mathcal{M}$ is a non-orientable surface of odd genus the number of triple points of j is odd, which implies that the cubical 4-polytope P has an odd number of facets. ☐

7.6 Symmetric templates

The three-dimension templates constructed above, which we call the *standard templates*, do not satisfy the conditions ($\mathrm{I}_3 3$) and ($\mathrm{I}_3 4$). In particular, the symmetry requirement ($\mathrm{I}_3 3$) is violated by the templates corresponding to the cases "empty", "single3", and "single6a," and it is satisfied by all others. For example, the standard template for "single5" is illustrated in Figure 29; it satisfies ($\mathrm{I}_3 3$) since there is only one diagonal symmetry hyperplane.

For the "empty" case an alternative template may be obtained from the cubical barycentric subdivision. The resulting cubification satisfies both conditions ($\mathrm{I}_3 3$) and ($\mathrm{I}_3 4$), and furthermore,

it has less faces — 96 facets, 149 vertices — than the standard template.
For the case “single3” an alternative cubification, of full symmetry, can be constructed from $\mathcal{C}''$ by truncating the lifted polytope corresponding to the lifted cubical ball $\mathcal{C}''$ by some additional hyperplanes.
For the case “single6a” we do not know how to get a cubification of full symmetry. This is the main obstacle for an extension of our constructions to higher dimensions (cf. Section 11).

8 An odd cubical 4-polytope with a dual Boy’s surface

Cubical 4-polytopes with odd numbers of facets exist by our Main Theorem 7.3. In this section we describe the construction of a cubical 4-polytope with an odd number of facets in more detail. The data for the corresponding model will be submitted to the `eg-models` archive.

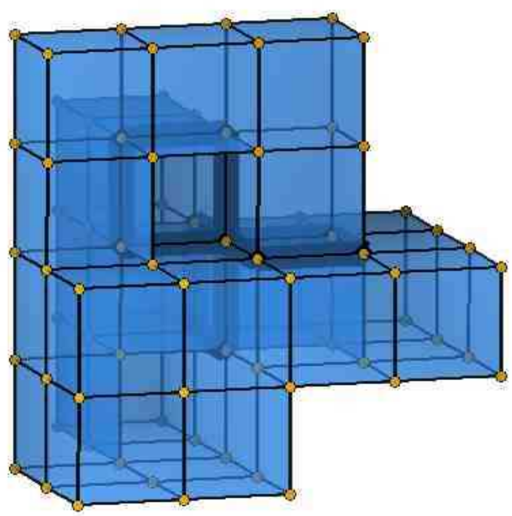
Theorem 8.1. *There is a cubical* 4*-polytope* P_{Boy} *with* f*-vector*

$$f \;=\; (17\,718, 50\,784, 49\,599, 16\,533)$$

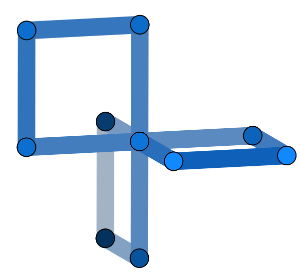
that has has a Boy surface as a dual manifold.

A grid immersion of Boy’s surface.

The construction starts with a grid immersion (cf. [25]) of Boy’s surface, that is, an immersion of the real projective plane with exactly one triple point and three double-intersection curves in a pattern of three loops [7] [18] [2]. This immersion $j : \mathcal{M} \looparrowright \mathbb{R}^3$ in shown in Figure 8.



(a) grid immersion



(b)

FIGURE 32: A grid immersion the Boy’s surface. Each double-intersection loop is of length four.

The 2-manifold $\mathcal{M}$ has the f-vector $f(\mathcal{M}) = (85, 168, 84)$, whereas the image of the grid immersion has the f-vector $f(j(\mathcal{M})) = (74, 156, 84)$. The vertex coordinates can be chosen such that the image $j(\mathcal{M})$ is contained in a pile of cubes $P_3(4, 4, 4)$.

A cubical 3-ball with a dual Boy's surface.

We apply Construction 6 to the grid immersion $j : \mathcal{M} \looparrowright \mathbb{R}^3$ to obtain a cubical 3-ball with a dual Boy's surface, and with an odd number of facets.

Since the image $j(\mathcal{M})$ is contained in a pile of cubes $P_3(4,4,4)$, the raw complex $\mathcal{A}$ given by Construction 6 is isomorphic to $P_3(5,5,5)$. Hence we have $5^3 - 74 = 51$ vertices of $\mathcal{A}$ that are not vertices of $j(\mathcal{M})$. We try to give an impression of the subdivision $\mathcal{C}^2$ of the 2-skeleton of $\mathcal{A}$ in Figure 33. The f-vector of $\mathcal{C}^2$ is $f = (4\,662,\ 9\,876,\ 5\,340)$.

The subdivision of the boundary of $\mathcal{A}$ consists of $150 = 6 \cdot 5 \cdot 5$ copies of the two-dimensional "empty pattern" template. Hence the subdivision of the boundary of $\mathcal{A}$ (given by $\mathcal{C}^2 \cap |\partial\mathcal{A}|$) has the f-vector $f = (1\,802,\ 3\,600,\ 1\,800)$.

The refinement $\mathcal{B}$ of $\mathcal{A}$ depends on templates that are used for the 3-cubes. We use the "symmetric" templates of Section 7.6. The f-vector of $\mathcal{B}$ is then $f = (15\,915,\ 45\,080,\ 43\,299,\ 14\,133)$. (The "standard set" of templates yields a cubical ball with 18 281 facets.)

Figure 34 illustrates the dual Boy's surface of the cubical 3-ball $\mathcal{B}$. It has the f-vector $f = (1\,998,\ 3\,994,\ 1\,997)$; its multiple-intersection loops have length 16. The ball $\mathcal{B}$ has 612 dual manifolds is total (339 of them without boundary).

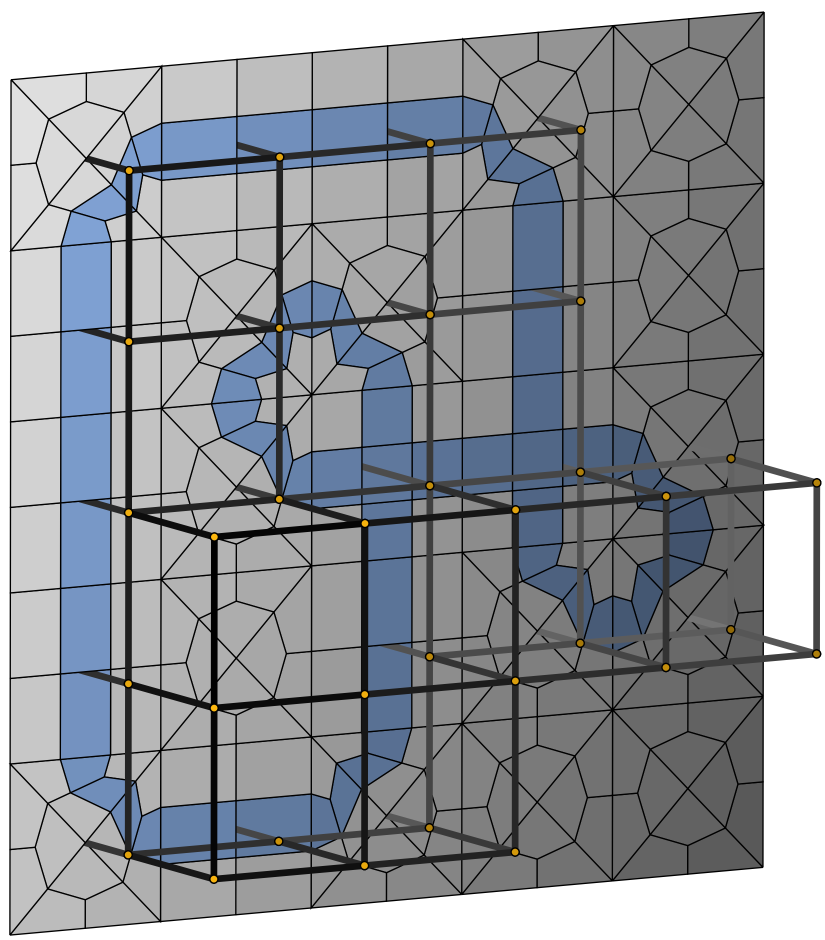


FIGURE 33: A sketch of the cubification of the 2-skeleton of $\mathcal{A}$.

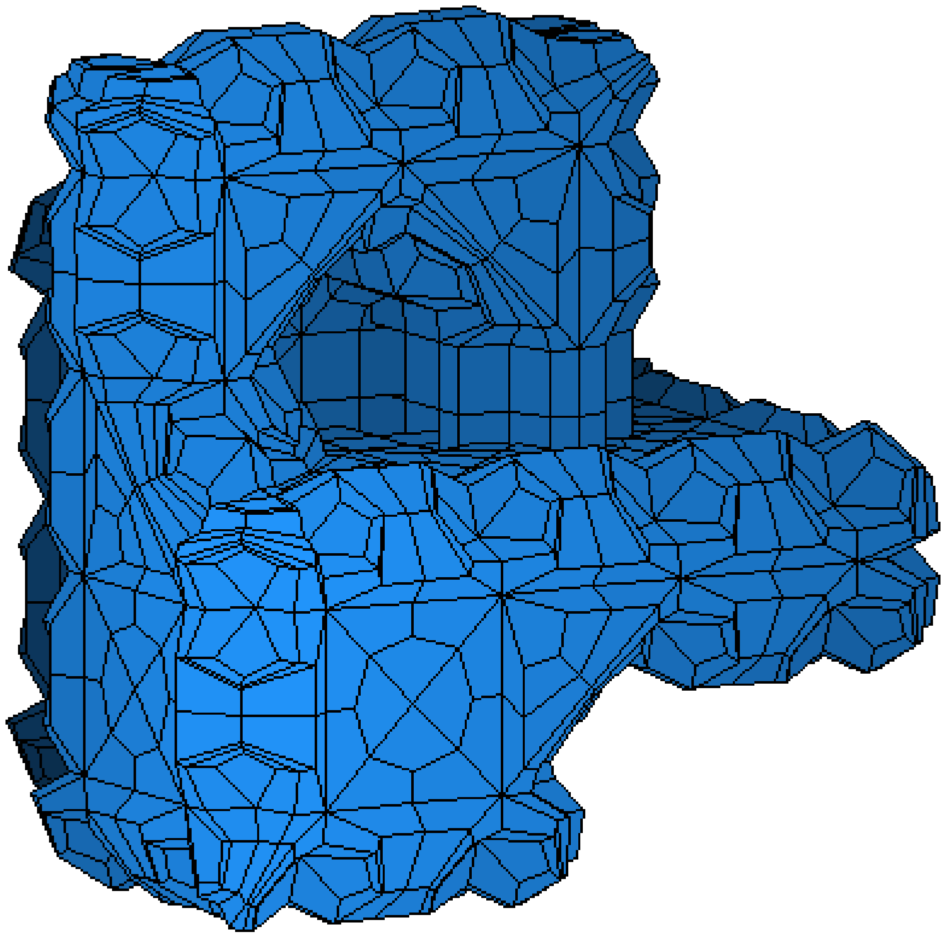


FIGURE 34: The dual Boy's surface of f-vector $f = (1\,998,\ 3\,994,\ 1\,997)$ of the cubical 3-ball $\mathcal{B}$.

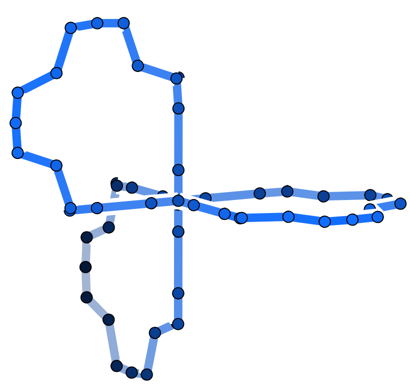


FIGURE 35: The multiple-intersection curve of the dual Boy's surface of the cubical 3-ball $\mathcal{B}$.

A cubical 4-polytope with a dual Boy's surface

A cubification $\mathcal{B}'$ of $\partial\mathcal{B}$ with an even number of facets is given by subdividing each facet of the raw ball $\mathcal{A}$ with a cubification for the empty pattern. Using the symmetric cubification for the empty pattern yields a regular cubical 3-ball $\mathcal{B}'$ with 12 000 facets. The lifted prism over $\mathcal{B}$ and $\mathcal{B}'$ yields a cubical 4-polytope with 27 933 facets.

However, using first stellar subdivisions, then a (simplicial) cone, and then cubical subdivisions on $\partial P_3(5,5,5)$, it is possible to produce a significantly smaller alternative cubification $\mathcal{B}''$ of $\partial\mathcal{B}$ with an even number of facets. Moreover, for this one can form the cone based directly on the boundary complex of $\mathcal{B}$ and thus "save the vertical part" of the prism. The resulting cubical 4-polytope P_{Boy} has $f_0 = 17\,718$ vertices and $f_3 = 16\,533$ facets. A further analysis of the dual manifolds of P_{Boy} shows that there are 613 dual manifolds in total: One dual Boy's surface of f-vector $f = (1\,998,\ 3\,994,\ 1\,997)$, one immersed surface of genus 20 (immersed with 104 triple points) with f-vector $(11\,470,\ 23\,016, 11\,508)$, and 611 embedded 2-spheres with various distinct f-vectors.

Verification of the instances

All the instances of the cubical 4-polytopes described above were constructed and verified as electronic geometry models in the polymake system by Gawrilow & Joswig [14], which is designed for the construction and analysis of convex polytopes. A number of our own tools for handling cubical complexes are involved as well. These cover creation, verification, and visualization of cubical complexes (for $d \in \{2,3\}$).

The instances are available from `http://www.math.tu-berlin.de/~schwartz/c4p`.

Whereas the construction of the instances involves new tools that were writted specifically for this purpose, the verification procedure uses only standard polymake tools. All tools used in the verification procedure are parts of polymake system which have been used (and thereby verified) by various users over the past years (using a rich variety of classes of polytopes).

The topology of the dual manifolds of our instances was examined using all the following tools:

- A homology calculation code based written by Heckenbach [16].
- The topaz module of the polymake project, which covers the construction and analysis of simplicial complexes.
- Our own tool for the calculation of the Euler characteristics.

9 Consequences

In this section we list a few immediate corollaries and consequences of our main theorem and of the constructions that lead to it. The proofs are quite immediate, so we do not give extended explanations here, but refer to [28] for details.

9.1 Lattice of f-vectors of cubical 4-polytopes

Babson & Chan [3] have obtained a characterizazion of the $\mathbb{Z}$-affine span of the f-vectors of cubical 3-spheres: With the existence of cubical 4-polytopes with an odd number of facets this extends to cubical 4-polytopes.

Corollary 9.1. *The $\mathbb{Z}$-affine span of the f-vectors (f_0, f_1, f_2, f_3) of the cubical 4-polytopes is characterized by*

(i) *integrality ($f_i \in \mathbb{Z}$ for all i),*
(ii) *the cubical Dehn-Sommerville equations $f_0 - f_1 + f_2 - f_3 = 0$ and $f_2 = 3f_3$, and*
(iii) *the extra condition $f_0 \equiv 0 \bmod 2$.*

Note that this includes modular conditions such as $f_2 \equiv 0 \bmod 3$, which are not "modulo 2." The main result of Babson & Chan [3] says that for cubical d-spheres and $(d+1)$-polytopes, $d \geq 2$, "all congruence conditions are modulo 2." However, this refers only to the modular conditions *which are not implied by integrality and the cubical Dehn-Sommerville equations*. The first example of such a condition is, for $d = 4$, the congruence (iii) due to Blind & Blind [6].

9.2 Cubical 4-polytopes with dual manifolds of prescribed genus

By our main theorem, from any embedding $M_g \rightarrow \mathbb{R}^3$ we obtain a cubical 4-polytope that has the orientable connected 2-manifold M_g of genus g as an embedded dual manifold. Indeed, this may for example be derived from a grid embedding of M_g into the pile of cubes $P(1, 3, 1 + 2g)$.

However, cubical 4-polytopes with an orientable dual manifold of prescribed genus can much more efficiently, and with more control on the topological data, be produced by means of connected sums of copies of the "neighborly cubical" 4-polytope C_4^5 with the graph of a 5-cube (compare Section 2.3).

Proposition 9.2. *For each $g > 0$, there is a cubical 4-polytope P_g with the following properties.*

(i) *The polytope P_g has exactly one embedded orientable dual 2-manifold $\mathcal{M}$ of genus g with f-vector $f(\mathcal{M}) = (12g + 4,\ 28g + 4,\ 14g + 2)$.*
(ii) *There is a facet F of P which is not intersected by the image of the dual manifold $\mathcal{M}$, and which is* affinely regular*, that is, there is an affine transformation between F and the standard cube $[-1, +1]^3$.*
(iii) *All other dual manifolds of P_g are embedded 2-spheres.*
(iv) *$f(P_g) = (24g + 8,\ 116g + 12,\ 138g + 6,\ 46g + 2)$.*

Taking now the connected sum of one example of a 4-polytope with a non-orientable dual 2-manifold, we obtain 4-polytopes with a non-orientable dual manifold of prescribed genus.

Corollary 9.3. *For each even $g > 0$, there is a cubical 4-polytope that has a cubation of the non-orientable connected 2-manifold M'_g of genus g as a dual manifold (immersed without triple points and with one double-intersection curve).*

For this, one can for example construct the 4-polytope associated with the grid immersion of the Klein bottle of f-vector $f = (52, 108, 56)$ as depicted in Figure 36.

Smaller cubical 4-polytopes with non-orientable cubical 4-polytopes can be produced by means of connected sums of the cubical 4-polytope P_{62} of Section 5 with a dual Klein bottle, and several copies of the neighborly cubical 4-polytope C_4^5. (Some "connector cubes" of Lemma 3.5 have to be used.) The resulting cubical 4-polytope has rather small f-vector entries, but the set of multiple-intersection points consists of five double-intersection curves.

Applying the same proof as above to the grid immersion of Boy's surface of the previous section yields the following result.

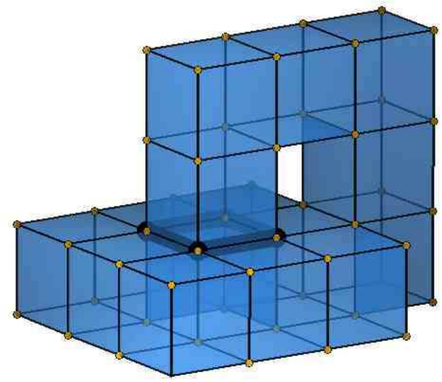


FIGURE 36: A grid immersion of the Klein bottle with one double-intersection curve and without triple points.

Corollary 9.4. *For each odd $g > 0$, there is a cubical 4-polytope that has a cubation of the non-orientable connected 2-manifold M'_g of genus g as a dual manifold (immersed with one triple point and three double-intersection curves of length 14).*

9.3 Higher-dimensional cubical polytopes with non-orientable dual manifolds

Corollary 9.5. *For each $d \geq 4$ there are cubical d-polytopes with non-orientable dual manifolds.*

Proof. By construction, the 4-dimensional instance P_{62} of Section 5 comes with a subdivision into a regular cubical 4-ball. Since one of its dual manifolds is not orientable, its 2-skeleton is not edge orientable, i. e. it contains a cubical Möbius strip with parallel inner edges. So if we now iterate the lifted prism construction of Section 4.1, then the resulting cubical d-polytopes ($d \geq 4$) will contain the 2-skeleton of P_{62}. By Proposition 2.1 they must also have non-orientable dual manifolds. □

10 Applications to hexa meshing

In the context of computer aided design (CAD) the surface of a workpiece (for instance a part of a car, ship or plane) is often modeled by a *surface mesh*. In order to analyze physical and technical properties of the mesh (and of the workpiece), finite element methods (FEM) are widely used.

Such a surface mesh is either a *topological mesh*, that is, 2-dimensional regular CW complex, or a *geometric mesh*, that is, a (pure) 2-dimensional polytopal complex cells. Common cell types of a surface mesh are triangles (2-simplices) and quadrangles. Thus a *geometric quad mesh* is a 2-dimensional cubical complex, and a *topological* one is a cubical 2-dimensional regular CW complex.

In recent years there has been growing interest in volume meshing. Tetrahedral volume meshes (simplicial 3-complexes) are well-understood, whereas there are interesting and challenging open questions both in theory and practice for hexahedral volume meshes, *hexa meshes* for short. That is, a *geometric hexa mesh* is a 3-dimensional cubical complex, and a *topological hexa mesh* is a cubical 3-dimensional regular CW complex.

A challenging open question in this context is whether each cubical quadrilateral geometric surface mesh with an even number of quadrangles admits a geometric hexa mesh. In our termi-

nology this problem asks whether each cubical PL 2-sphere with an even number of facets admits a cubification. Thurston [31] and Mitchell [23] proved independently that every topological quad mesh with an even number of quadrangles admits a topological hexa mesh. Furthermore, Eppstein showed in [10] that a linear number of topological cubes suffices, and Bern, Eppstein & Erickson proved the existence of a (pseudo-)shellable topological hexa mesh [5].

10.1 Parity change

Another interesting question deals with the parity of the number of facets of a mesh. For quad meshes there are several known parity changing operations, that is, operations that change the numbers of facets without changing the boundary. In [5], Bern, Eppstein & Erickson raised the following questions:

(i) Are there geometric quad meshes with geometric hexa meshes of both parities?
(ii) Is there a *parity changing operation* for geometric hexa meshes, which would change the parity of the number of facets of a cubical 3-ball without changing the boundary?

From the existence of a cubical 4-polytope with odd number of facets we obtain positive answers to these questions.

Corollary 10.1.

(i) *Every combinatorial* 3*-cube has a cubification with an even number of facets. Furthermore, this cubification is regular and even Schlegel.*
(ii) *Every combinatorial* 3*-cube is a facet of a cubical* 4*-polytope with an odd number of facets.*
(iii) *There is a parity changing operation for geometric hexa meshes.*

Proof. For (ii) let F be a combinatorial 3-cube and P a cubical 4-polytope with an odd number of facets. By Lemma 3.5 there is a combinatorial 4-cube C that has both F and a projectively regular 3-cube G as facets. Let F' be an arbitrary facet of P. Then there is a combinatorial 4-cube C' that has both F' and a projectively regular 3-cube G' as facets. Then the connected sum of P and C based on the facet F' yields a cubical 4-polytope P' with an odd number of facets, and with a projectively regular 3-cube G'' as a facet. The connected sum of P' and C glueing the facets G and G'' yields a cubical 4-polytope with an odd number of facets, and with a projective copy of F as a facet.

The statements (i) and (iii) follow from (ii) via Schlegel diagrams. □

10.2 Flip graph connectivity

In analogy to the concept of flips for simplicial (pseudo-)manifolds one can define *cubical flips* for quad or hexa meshes; compare [5]. In the meshing terminology the flip graph is defined as follows. For any domain with boundary mesh, and a type of mesh to use for that domain, define the *flip graph* to be a graph with (infinitely many) vertices corresponding to possible meshes of the domain, and an edge connecting two vertices whenever the corresponding two meshes can be transformed into each other by a single flip.

In this framework, the question concerning a parity changing operation can be phrased as asking for a description of the connected components of the flip graph. As an immediate consequence of the corollary above we obtain the following result.

Corollary 10.2. *For every geometric hexa mesh the cubical flip graph has at least two connected components.*

11 The next step

In this paper we are primarily concerned with the realization of 2-manifold immersions in terms of cubical 4-polytopes, but the higher-dimensional cases are interesting as well. For example, one would like to know whether there are cubical 5-polytopes with an odd number of facets. (There are *no* such d-polytopes for $d = 6$, or for $8 \leq d \leq 13$; see the [3, Sect. 7].) For this we have to realize a normal crossing immersion of 3-manifold into S^4 by a cubical 5-polytope with an odd number of quadruple points. Such immersions exist by an abstract result of Freedman [13] [1], but more concretely by John Sullivan's observation (personal communication) that there are regular sphere eversions of the 2-sphere with exactly one quadruple point [30] [12] and from any such one obtains a normal-crossing immersion $S^3 \looparrowright S^4$ with a single quadruple point.

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The illustrations of polytopes and balls were produced in the polymake system of Gawrilow & Joswig [14], via javaview of Polthier et. al. [26]. (A small number of figures was drawn with xfig.)

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